Very Mad Families

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ABSTRACT. The notion of very mad family is a strengthening of the notion of mad family of functions. Here we show existence of very mad families in different contexts.

1. Introduction

Almost disjoint families and maximal almost disjoint families have received a lot of attention in set theory. In their study many different varieties have been introduced. There are the "standard" almost disjoint families, and varieties on different spaces and with different additional conditions.

The study of similarities and differences among these different varieties leads to many interesting results and questions.

DEFINITION 1.1. We call $x, y \subseteq \mathbb{N}$ *almost disjoint* iff both are infinite and $x \cap y$ is finite.

A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is a *maximal almost disjoint family (of subsets of the natural numbers)* iff it is infinite, consists of pairwise almost disjoint sets, and is not properly contained in another such family.

We call $f, g \in {}^{\mathbb{N}}\mathbb{N}$ *almost disjoint* or *eventually different* iff the set $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite. (${}^{\mathbb{N}}\mathbb{N}$ is the *Baire space* of functions $\mathbb{N} \to \mathbb{N}$ with the product topology obtained from the discrete topology on $\mathbb{N}$).

Let $\mathrm{Sym}(\mathbb{N}) \subseteq {}^{\mathbb{N}}\mathbb{N}$ denote the group of bijections from the natural numbers to the natural numbers with composition as the group operation.

A family of permutations $\mathcal{A} \subseteq \mathrm{Sym}(\mathbb{N})$ is *almost disjoint* iff all distinct $f, g \in \mathcal{A}$ are almost disjoint. It is a *maximal almost disjoint family (of permutations)* iff it is almost disjoint and not properly included in another such family.

A family $\mathcal{A} \subseteq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* iff it is an almost disjoint family and it is a subgroup of $\mathrm{Sym}(\mathbb{N})$. It is a *maximal cofinitary group* iff it is a cofinitary group not properly contained in another cofinitary group.

Define the cardinal a to be the least cardinality of a maximal almost disjoint family of subsets of $\mathbb{N}$, a_p to be the least cardinality of a maximal almost disjoint family of permutations, and similarly, a_g to be the least cardinality of a maximal cofinitary group.

Although these notions of almost disjointness and the associated cardinals are closely related there are essential differences – for example from [**BSZ**] we have the following theorem.

THEOREM 1.2 ([**BSZ**]). *It is consistent with* ZFC *that* $\mathsf{a} < \mathsf{a}_p = \mathsf{a}_g$.

So the difference in which space the almost disjoint family is defined on can be used to get them to be of different cardinality (in [**YZ2**] it was proved that $\mathsf{a} < \mathsf{a}_p$ is consistent, and in [**HSZ**] that $\mathsf{a} < \mathsf{a}_g$ is consistent). The following is a question is however still open.

QUESTION 1.3 ([**YZ2**]). *Is it consistent with* ZFC *that* a_p *is distinct from* a_g*?*

That is, does the group structure influence the possible cardinalities of the almost disjoint family?

Another question that has been answered for some of these families, but certainly not all of them is how definable they can be.

The following are some well known theorems related to this question.

THEOREM 1.4 ([**ARDM**]). *A maximal almost disjoint subfamily of* $\mathcal{P}(\mathbb{N})$ *cannot be Borel.*

THEOREM 1.5 ([**AM**]). *Assuming the axiom of constructibility there exists a coanalytic maximal almost disjoint subfamily of* $\mathcal{P}(\mathbb{N})$.

Juris Steprāns introduced a strengthening of the notion of mad family of functions (strongly mad family) having the right properties to show that there do not exist analytic ($\mathbf{\Sigma^1_1}$) strongly mad families. This motivated us to study first their existence and to show with Yi Zhang a companion result to Steprāns' result — that the axiom of constructibility implies that there are coanalytic ($\mathbf{\Pi^1_1}$) strongly mad families. Steprāns' result and our companion result appear in the paper [**KSZ**].

In working on the existence of strongly mad families it quickly became apparent that the results go through for a natural further strengthening of mad families, which we call very mad.

DEFINITION 1.6. A function $f \in {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by a family $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff there exist $f_0, \ldots, f_n \in \mathcal{A}$ such that $f \setminus \bigcup_{i \leq n} f_i$ is finite.

A family $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by a family $\mathcal{A}$ iff there exists an $f \in F$ that is finitely covered by $\mathcal{A}$ (note: only one function in the family needs to be finitely covered).

A family $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *strongly mad family* if it is an almost disjoint family and for every countable $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ that is not finitely covered by $\mathcal{A}$ there is a function $g \in \mathcal{A}$ such that for all $f \in F$ the intersection $f \cap g$ is infinite.

A family $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *very mad family* if it is an almost disjoint family and for every family $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ that is not finitely covered by $\mathcal{A}$ such that $|F| < |\mathcal{A}|$ there is a function $g \in \mathcal{A}$ such that for all $f \in F$ the intersection $f \cap g$ is infinite.

Note the following about these notions:

- Any very mad family is strongly mad (there do not exist countable mad families in Baire space).
- Any strongly mad family is mad (a function almost disjoint from the family would not be finitely covered).
- A strongly mad family of cardinality $\aleph_1$ is very mad.

- Under the continuum hypothesis the notions of very mad and strongly mad coincide.

In this paper we work on the existence question for very mad families. We prove below that

THEOREM 1.7. *Martin's Axiom implies that very mad families exist and are of cardinality* $\mathfrak{c}$.

THEOREM 1.8. *Any model of* $\mathsf{ZFC}+\mathsf{CH}$ *contains a very mad family that is still very mad in any Cohen extension of that model.*

THEOREM 1.9. *For any model of* ZFC *in which there is a regular uncountable cardinal* κ *less than the continuum, there exists a forcing extension in which the size of the continuum doesn't change, and there is a very mad family of cardinality* κ.

In working on these existence results we also found the counterintuitive, to us, result that there is in fact much "room" outside a very mad family.

DEFINITION 1.10. Two families $\mathcal{A}, \mathcal{B} \subseteq {}^{\mathbb{N}}\mathbb{N}$ are *orthogonal* iff no member of either family is finitely covered by the other family.

THEOREM 1.11. *Martin's axiom implies that there exists a collection of continuum many very mad families that are pairwise orthogonal.*

Some questions that remain after this work are the following.

QUESTION 1.12. *Is the existence of very mad or strongly mad families a theorem of* ZFC*?*

And because all of our constructions give very mad families of regular size we get the following.

QUESTION 1.13. *Is it consistent that there exist very mad or strongly mad families of singular size? Of countable cofinality?*

2. Martin's Axiom

In this section we prove that MA suffices to prove the existence of very mad families.

We define for any subset $\mathcal{A}$ of Baire space a notion of forcing $\mathbb{P}_{\mathcal{A}}$ (which will also be used in section 4). It consists of all conditions of the form $\langle s, A\rangle$, such that

- s is a finite partial function $\mathbb{N} \rightharpoonup \mathbb{N}$, and
- A is a finite subset of $\mathcal{A}$.

The ordering[1] $\langle s_2, A_2\rangle \leq \langle s_1, A_1\rangle$ is defined by

$$s_1 \subseteq s_2 \ \wedge\ A_1 \subseteq A_2 \ \wedge\ \forall f \in A_1\ [f \cap s_2 \subseteq s_1].$$

This notion of forcing is σ-centered (a strengthening of c.c.c.) since there are only countably many choices for s in a condition and any two conditions with identical first coordinate are compatible. This shows that in fact for the following Lemma and Theorem the hypothesis $\mathfrak{p} = \mathfrak{c}$ suffices, since this equality implies MA(σ-centered).

[1] $q \leq p$ means that q is an extension of p.

Since CH implies MA the results below are also true under CH. Of course in the case of CH you can do essentially the same construction while avoiding the mention of a notion of forcing.

LEMMA 2.1 (MA). *Assume that $\mathcal{A}$ is an almost disjoint family of functions with $|\mathcal{A}| < 2^{\aleph_0}$ and that F is a family not finitely covered by $\mathcal{A}$ with $|F| < 2^{\aleph_0}$. Then there exists a function g such that:*

- *$\mathcal{A} \cup \{g\}$ is an almost disjoint family of functions, and*
- *for all $f \in F$ the set $f \cap g$ is infinite.*

PROOF. For each $f \in F$, $g \in \mathcal{A}$ and $n \in \mathbb{N}$ let

- $C_g := \{\langle s, A\rangle : g \in A\}$,
- $D_n := \{\langle s, A\rangle : n \in \mathrm{dom}(s)\}$,
- $E_{f,n} := \{\langle s, A\rangle : \exists m \geq n\ f(m) = s(m)\}$.

All these sets are dense: Let $\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}}$. For C_g we have $\langle s, A \cup \{g\}\rangle \leq \langle s, A\rangle$ and $\langle s, A \cup \{g\}\rangle \in C_g$. For D_n we need to find an extension of s such that n is in its domain (if this is not already the case). Since A is finite, there exists an m such that $m \neq h(n)$ for all $h \in A$ and we take $s \cup \{(n, m)\}$. For $E_{f,n}$ note that since f is not finitely covered by $\mathcal{A}$, there is an $m > n$ such that $f(m) \notin \{g(m) : g \in A\}$. Then $\langle s \cup \{(m, f(m))\}, A\rangle \leq \langle s, A\rangle$ and $\langle s \cup \{(m, f(m))\}, A\rangle \in E_{f,n}$.

The family

$$\mathcal{D} = \{C_g : g \in \mathcal{A}\} \cup \{D_n : n \in \mathbb{N}\} \cup \{E_{f,n} : f \in F, n \in \mathbb{N}\},$$

has cardinality less than $2^{\aleph_0}$. Therefore by MA there is a filter G meeting all of the sets in $\mathcal{D}$. The function $g = \bigcup\{s : \exists A \subseteq \mathcal{A}\ [\langle s, A\rangle \in G]\}$ is then the desired function. $\square$

THEOREM 2.2 (MA). *There exists a very mad family, and any such family is of cardinality $2^{\aleph_0}$.*

PROOF. We will construct functions g_β such that $\{g_\alpha : \alpha < 2^{\aleph_0}\}$ is a very mad family of size $2^{\aleph_0}$. Let $\{f_\alpha : \alpha < 2^{\aleph_0}\}$ be an enumeration of ${}^{\mathbb{N}}\mathbb{N}$. At stage β we do the following:

We let F be the maximal subset of $\{f_\alpha : \alpha < \beta\}$ that is not finitely covered by $A_\beta := \{g_\alpha : \alpha < \beta\}$ (F is the set of functions in $\{f_\alpha : \alpha < \beta\}$ that are not finitely covered by A_β). By Lemma 2.1 there exists a g_β such that $A_\beta \cup \{g_\beta\}$ is an almost disjoint family of functions, and for any $f \in F$ the set $f \cap g_\beta$ is infinite (if the set F is empty, we just get a new function almost disjoint from all of A_α).

Now let $\mathcal{A} = \{g_\beta : \beta < 2^{\aleph_0}\}$; we claim that $\mathcal{A}$ is a very mad family. Suppose not. Then there is an F such that F is not finitely covered by $\mathcal{A}$ and $|F| < |\mathcal{A}| = 2^{\aleph_0}$, and there is no $g \in \mathcal{A}$ that meets every member of F infinitely often. But since this F is of cardinality less than $2^{\aleph_0}$ we have $F \subseteq \{f_\alpha : \alpha < \beta\}$ for some $\beta < 2^{\aleph_0}$ (MA implies that $2^{\aleph_0}$ is regular), but then g_β will meet all members of F infinitely often, which is a contradiction.

The iteration step, basically Lemma 2.1, proves the second part of the theorem. $\square$

3. In the Cohen Model

We prove that in any model of the continuum hypothesis there is a very mad family that survives Cohen forcing. For this we need the following lemma, which is from [**K**, Lemma 2.2, p. 256].

We let $\mathrm{Fn}(I,2)$ denote the set of finite partial functions $I \rightharpoonup 2$ ordered by reverse inclusion.

LEMMA 3.1. *Suppose $I, S \in M$. Let G be $\mathrm{Fn}(I,2)$-generic over M, and let $X \subseteq S$ with $X \in M[G]$. Then $X \in M[G \cap \mathrm{Fn}(I_0,2)]$ for some $I_0 \subseteq I$ such that $I_0 \in M$ and $(|I_0| \le |S|)^M$.*

THEOREM 3.2. *Let M be a model of the continuum hypothesis and $I \in M$. Then there is a very mad family $\mathcal{A}$ in M such that for any $\mathrm{Fn}(I,2)$-generic set G, $M[G] \models$ "$\check{A}$ is a very mad family of size $\aleph_1$".*

PROOF. We construct a very mad family that survives forcing with $\mathrm{Fn}(\mathbb{N},2)$, and then show that this family survives forcing with $\mathrm{Fn}(I,2)$ for any I. Note that since the continuum hypothesis is true in M, strongly and very mad families in M are the same thing, and that the very mad family in M will be of size $\aleph_1$, so it only has to contain functions capturing any countable collection in the extension.

Since

$$\begin{aligned}
|\,\mathrm{Fn}(\mathbb{N},2) \times \{\tau : \tau \text{ is a nice name for a subset of } (\mathbb{N}\times\mathbb{N})\check{}\ \}| \\
\le \aleph_0 \times (\#\text{ anti-chains in } \mathrm{Fn}(\mathbb{N},2)) \times |\mathbb{N}\times\mathbb{N}| \\
\le \aleph_0 \times 2^{\aleph_0} \times \aleph_0 \overset{\mathsf{CH}}{=} \aleph_1,
\end{aligned}$$

we can enumerate $\mathrm{Fn}(\mathbb{N},2) \times \{\tau : \tau \text{ is a nice name for a subset of } (\mathbb{N}\times\mathbb{N})\check{}\ \}$ as $\langle (p_i,\tau_i) : i < \omega_1\rangle$.

We construct $\mathcal{A} = \{g_\alpha : \alpha < \omega_1\}$, the very mad family, recursively.

Assume all g_α for $\alpha < \beta$ have been defined. We need g_β to satisfy:

(G1) for all $\alpha < \beta$ the functions g_α and g_β are almost disjoint, and

(G2) if $F_\beta := \{\tau_\alpha : \alpha < \beta, p_\beta \Vdash$ "τ_α is a total function and τ_α is not finitely covered by $\{\check{g}_\gamma : \check{\gamma} < \check{\beta}\}$"$\}$, then $(\forall \tau_\alpha \in F_\beta)\ p_\beta \Vdash$ "$|\tau_\alpha \cap \check{g}_\beta| = \check{\omega}$". $p_\beta \Vdash$ "$|\tau_\alpha \cap \check{g}_\beta| = \check{\omega}$" is equivalent to
$$(\forall n)(\forall q \le p_\beta)(\exists r \le q)(\exists m \ge n)\ r \Vdash \text{“}\tau_\alpha(\check{m}) = \check{g}_\beta(\check{m})\text{”}.$$

Enumerate $\mathbb{N} \times \{q : q \le p_\beta\}$ as $\langle (n_i,q_i) : i < \omega\rangle$, $\langle g_\alpha : \alpha < \beta\rangle$ as $\langle g'_i : i < \omega\rangle$ and F_β as $\langle \tau'_i : i < \omega\rangle$.

Recursively define g_β. Before stage s we have g_β defined on $\{0,\dots,n_s\}$. At stage s we want to define g_β on $\{n_s+1,\dots,n_{s+1}\}$ for some n_{s+1} so that G1 and G2 will eventually be satisfied.

The requirements at this stage will be:

(L1) $\forall n \in \{n_s+1,\dots,n_{s+1}\}\ (g_\beta(n) \notin \{g'_0(n),\dots,g'_s(n)\})$, and

(L2) there are $n_{s,0},\dots,n_{s,s} > n_s$ all distinct and $r_{s,0},\dots,r_{s,s} \le q_s$ such that for all i from 0 to s we have $r_{s,i} \Vdash$ "$\tau'_i(\check{n}_{s,i}) = \check{g}_\beta(\check{n}_{s,i})$".

Requirement L1 ensures that g_β will satisfy G1 and requirement L2 ensures that g_β will satisfy G2.

If $p_\beta \Vdash$ "τ'_i is a total function not covered by $\check{g}'_0,\dots,\check{g}'_s$", then for every $m > n_s$ there is an $n_{s,i} > m$ and $r_{s,i} \le q_s$ such that $r_{s,i} \Vdash$ "$\tau'_j(\check{n}_{s,i}) \notin \{\check{g}'_i(\check{n}_{s,i}) : 0 \le i \le s\}$". Then below $r_{s,i}$ there is a condition $r'_{s,i}$ that decides the value of $\tau'_i(\check{n}_{s,i})$.

We use this observation repeatedly to find $n_{s,0} < \cdots < n_{s,s}$ all larger than n_s, and define $g_\beta(n_{s,i})$ to be the number that $\tau'_i(\check{n}_{s,i})$ is forced to be by $r'_{s,i}$. Then we set $n_{s+1} = n_{s,s}$ and set $g_\beta(n)$, for $n < n_{s+1}$ such that $g_\beta(n)$ is not defined yet, to be any number not in $\{g'_i(n) : 0 \leq i \leq s\}$. This completes the construction.

We show that $\mathcal{A} = \{g_i : i < \omega_1\}$ is a very mad family in the forcing extension by $\mathrm{Fn}(\mathbb{N}, 2)$. For this let G be $\mathrm{Fn}(\mathbb{N}, 2)$-generic over M. First note $\mathcal{A}$ is an almost disjoint family in $M[G]$, since it is almost disjoint in M: the functions are almost disjoint by the first requirement. To see it is very mad let F be a countable family of functions, all of which are not finitely covered by $\check{A}$. Then in $M[G]$ we have $F \subseteq \{\tau_i[G] : i < \omega_1\}$; therefore there exists an α (by the countability of F) such that $F \subseteq \{\tau_i[G] : i < \alpha\}$, and this is forced by some $p \in G$. Then at some point β in the construction (after stage α) when we have a p_β equal to p we will correctly deal with a superset of F (and therefore with F), by the second requirement.

It remains to show that $\mathcal{A}$ is very mad in the forcing extension by $\mathrm{Fn}(I, 2)$ for any I. Suppose that in some forcing extension by $\mathrm{Fn}(I, 2)$ the family $\mathcal{A}$ defined above is no longer very mad. There is then a countable family F of functions not finitely covered by $\mathcal{A}$ for which there does not exist a $g \in \mathcal{A}$ hitting all of them infinitely often. Each function $f \in F$ appears in some $\mathrm{Fn}(I_f, 2)$, with $I_f \subseteq I$ countable (by Lemma 3.1), so all of them appear in the forcing extension by $\mathrm{Fn}(\bigcup_{f \in F} I_f, 2) \cong \mathrm{Fn}(\mathbb{N}, 2)$. The above argument for $\mathrm{Fn}(\mathbb{N}, 2)$ shows there is a function $g \in \mathcal{A}$ that hits all of the functions in F infinitely often. This contradicts the existence of such a family F. $\square$

4. A Very MAD Family not of size $\mathfrak{c}$

In this section we show that consistently there exist very mad families that are not of the same size as the continuum. The forcing is based on the proof of the similar result for maximal cofinitary groups by Yi Zhang in [**YZ3**].

THEOREM 4.1. *Let M be a model of* ZFC *and in M there is a regular cardinal κ such that in M $\aleph_1 \leq \kappa < 2^{\aleph_0} = \lambda = \kappa^{\aleph_0}$. Then there exists a c.c.c. forcing $\mathbb{P}$ such that $M^{\mathbb{P}}$ satisfies*

(1) $2^{\aleph_0} = \lambda$.
(2) *There exists a very mad family $\mathcal{A}$ of cardinality κ.*

In the proof we will use the c.c.c. poset $\mathbb{P}_{\mathcal{A}}$ from section 2. First we prove its main property (which is basically Lemma 2.1 rephrased in the language of forcing).

LEMMA 4.2. *If N is a model of* ZFC, *$f \in N \cap {}^{\mathbb{N}}\mathbb{N}$, $\mathcal{A} \subset {}^{\mathbb{N}}\mathbb{N}$, $\mathcal{A} \in N$ and f is not finitely covered by $\mathcal{A}$, then the generic function g obtained from forcing with $\mathbb{P}_{\mathcal{A}}$ over N satisfies*

- *$\mathcal{A} \cup \{g\}$ is an almost disjoint family,*
- *$f \cap g$ is infinite.*

PROOF. We use the notation from the proof of Lemma 2.1.

g is a total function since for every $n \in \mathbb{N}$ $D_n = \{\langle s, H\rangle : n \in \mathrm{dom}(s)\}$ is dense and in N. $\mathcal{A} \cup \{g\}$ is almost disjoint since for every $g' \in \mathcal{A}$ the set $C_{g'} = \{\langle s, H\rangle : g' \in H\}$ is dense and in N. $f \cap g$ is infinite as for every $n \in \mathbb{N}$ the set $E_{f,n} = \{\langle s, H\rangle : \exists m \geq n \; m \in \mathrm{dom}(s) \wedge s(m) = f(m)\}$ is dense (by not finitely covering) and in N. $\square$

Proof of Theorem 4.1. From this lemma we get that if we force with $\mathbb{P}_{\mathcal{A}}$ we get an a.d. family $\mathcal{A} \cup \{g\}$ where there is a function, g, that hits infinitely often all functions in the ground model that are not finitely covered by $\mathcal{A}$.

$\mathbb{P}$ is the κ step finite support iteration of the $\mathbb{P}_{\mathcal{A}}$, where $\mathcal{A}$ at step α consists of the generics added so far ($\mathcal{A}_\alpha = \{g_\beta : \beta < \alpha\}$).

For any α, $|\mathcal{A}_\alpha| = |\alpha|$. So we can use as underlying set for $\mathbb{P}_{\mathcal{A}_\alpha}$ the set of $\langle s, H\rangle$, $s : \mathbb{N} \rightharpoonup \mathbb{N}$ finite partial and $H \subseteq |\alpha|$ ($= |\mathcal{A}|$) finite. This shows $|\mathbb{P}_{\mathcal{A}_\alpha}| = \max\{\omega, |\alpha|\}$, from which we get $|\mathbb{P}| = \kappa$: since we use a finite support iteration of length κ we can use finite subsets of κ to indicate at which indices the element is non-maximal, and then for each of those an element from $\bigcup_{\alpha<\kappa} \mathbb{P}_{\mathcal{A}_\alpha}$ which is in the right coordinate (this also shows the poset can be taken to have the underlying set in the ground model).

Since $\mathbb{P}$ is a finite support iteration of c.c.c. posets it preserves cardinals. And using the c.c.c., $|\mathbb{P}| = \kappa$ and $\kappa^\omega = 2^{\aleph_0}$ we see that in the forcing extension $2^{\aleph_0}$ is still λ.

Now we see that the resulting collection $\mathcal{A} = \{g_\alpha : \alpha < \kappa\}$ is very mad as follows.

Let F, $|F| < |\mathcal{A}| = \kappa$ be a collection of functions not finitely covered by $\mathcal{A}$ (so also not finitely covered by any subcollection of $\mathcal{A}$). Since the forcing is finite support and $|F| < \mathrm{cf}(\kappa) = \kappa$, F already appears at some stage before κ. Then the generic added at any later stage will hit all members of F infinitely often (by the lemma). □

5. Orthogonal Very MAD Families

In this section we'll prove the existence, under Martin's axiom, of many orthogonal very mad families. This intuitively shows that there can be much "room" outside of a very mad family.

Let $\langle \mathcal{A}_\alpha : \alpha < \beta\rangle$ be subsets of Baire space. Define the notion of forcing $\mathbb{Q}_{\langle \mathcal{A}_\alpha : \alpha < \beta\rangle}$, to consist of all conditions $\bar{p}$ that are finite partial functions with domain contained in β and $\bar{p}(\alpha) \in \mathbb{P}_{\mathcal{A}_\alpha}$. If $\bar{p}(\alpha) = \langle s, A\rangle$ we define $\pi_0(\bar{p}(\alpha)) := s$ and $\pi_1(\bar{p}(\alpha)) := A$. Define $\bar{q} \leq \bar{p}$ iff $\forall \alpha \in \mathrm{dom}(\bar{p})$ $[\bar{q}(\alpha) \leq_{\mathbb{P}_{\mathcal{A}_\alpha}} \bar{p}(\alpha)]$.

Lemma 5.1 (MA). *Let $\mathcal{A}_\alpha$ ($\alpha < \beta < \mathfrak{c}$) be almost disjoint families, that are pairwise orthogonal, and let F_α ($\alpha < \beta$) be families of functions such that $|F_\alpha| < 2^{\aleph_0}$ and F_α is not finitely covered by $\mathcal{A}_\alpha$. Then there exist functions g_α ($\alpha < \beta$) such that*

- *for all $\alpha < \beta$, the family $\mathcal{A}_\alpha \cup \{g_\alpha\}$ is an almost disjoint family, and*
- *for all $\alpha < \beta$ and $f \in F_\alpha$, the set $f \cap g_\alpha$ is infinite, and*
- *for all $\alpha_1, \alpha_2 < \beta$, the families $A_{\alpha_1} \cup \{g_{\alpha_1}\}$ and $A_{\alpha_2} \cup \{g_{\alpha_2}\}$ are orthogonal.*

Proof. In addition to the dense sets used in the proof of Lemma 2.1 for each coordinate $\alpha < \beta$, define for all $\alpha_1 \neq \alpha_2 < \beta, n \in \mathbb{N}$ and $a \in [\mathcal{A}_{\alpha_2}]^{<\omega}$ the sets

$$A_{\alpha_1,\alpha_2,a,n} := \{\bar{p} : a \subseteq \pi_1(\bar{p}(\alpha_2)) \ \wedge \exists m > n\big[m \in \mathrm{dom}(\pi_0(\bar{p}(\alpha_1))) \ \wedge \\ \pi_0(\bar{p}(\alpha_1))(m) \notin \{f(m) : f \in a\}\big]\}.$$

These sets are easily seen to be dense, and combined with the dense sets from Lemma 2.1 there are still fewer than continuum many. So MA implies there is a filter meeting them all.

The dense sets from Lemma 2.1 guarantee the first two items of the theorem, and the new dense sets $A_{\alpha_1,\alpha_2,a,n}$ ensure that the resulting families are still orthogonal: the filter intersecting all dense sets $A_{\alpha_1,\alpha_2,a,n}$ for $n \in \mathbb{N}$ ensures that g_{α_1} is not finitely covered by a. $\square$

THEOREM 5.2 (MA). *There exist very mad families $\mathcal{A}_\alpha$ ($\alpha < \mathfrak{c}$) such that for all $\alpha_1 \neq \alpha_2 < \mathfrak{c}$ the families $\mathcal{A}_{\alpha_1}$ and $\mathcal{A}_{\alpha_2}$ are orthogonal.*

PROOF. We will construct the families $\mathcal{A}_\alpha$ recursively as $\mathcal{A}_\alpha := \bigcup_{\gamma<\mathfrak{c}} \mathcal{A}_{\alpha,\gamma}$.

We start with $\langle \mathcal{A}_{\alpha,0} := \emptyset : \alpha < \mathfrak{c}\rangle$. Then at step β we apply Lemma 5.1 with $\mathbb{Q}_{\langle \mathcal{A}_{\alpha,\beta}:\alpha<\beta\rangle}$ to obtain $\langle g_\alpha : \alpha < \beta\rangle$ and set $\mathcal{A}_{\alpha,\beta+1} := \mathcal{A}_{\alpha,\beta} \cup \{g_\alpha\}$ for $\alpha < \beta$, and $\mathcal{A}_{\alpha,\beta+1} := \mathcal{A}_{\alpha,\beta}$ for $\beta \leq \alpha < \mathfrak{c}$. $\square$

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