Cofinitary Groups and Other Almost Disjoint Families of Reals

by

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CHAPTER I

Introduction

Almost disjoint families and maximal almost disjoint (mad) families have received a lot of attention in set theory. In their study many different varieties have been introduced. There are the "standard" almost disjoint families, and varieties on different spaces and with different additional conditions.

The general definitions of the notions almost disjoint, almost disjoint family, and maximal almost disjoint family are as follows. Take a collection X consisting of objects of a certain (infinite) cardinality κ. Two objects x, y in X are called *almost disjoint* if the intersection of x and y, $x \cap y$, is of cardinality less than κ. A subset $\mathcal{A}$ of X is an *almost disjoint family* if every two distinct objects in $\mathcal{A}$ are almost disjoint. Such a subset $\mathcal{A}$ is a *maximal almost disjoint family* if $\mathcal{A}$ is an almost disjoint family and there does not exist an almost disjoint family $\mathcal{B}$ of X such that $\mathcal{A}$ is a proper subset of $\mathcal{B}$ ($\mathcal{A}$ is maximal with respect to inclusion).

Existence of maximal almost disjoint families is easy to see. Using Zorn's lemma (an equivalent of the axiom of choice) and noticing that the union of an increasing chain (with respect to inclusion) of almost disjoint families is almost disjoint, we see that there does exist a maximal almost disjoint family.

Different notions of almost disjointness can be formed from the general notion in different ways — we will mention the two we will use. One can change the set X of which the almost disjoint families are subsets. Usual choices for this are $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers, or ${}^{\mathbb{N}}\mathbb{N}$, the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Other choices for X have been and are receiving attention, for instance see [6]. Another way to change the notion of almost disjointness is by imposing additional conditions on the family $\mathcal{A}$. One can for instance impose a group structure on $\mathcal{A}$ if there is one on X. With additional conditions imposed, the existence of maximal almost disjoint families needs to be reinvestigated.

In this thesis we will study some different varieties of almost disjoint and maximal almost disjoint families (for all of them κ as in the above general definition of almost disjointness will be equal to $\aleph_0$). Some of the results and questions are inspired by similarities and differences among these different varieties of almost disjoint families.

1.1 Subsets of $\mathbb{N}$

The standard example of almost disjointness is the following.

Definition I.1. We call $x, y \subseteq \mathbb{N}$ *almost disjoint* if both are infinite and $x \cap y$ is finite.

For various reasons, which we will explain shortly, we impose the additional condition that the family is infinite on the related notion of maximal almost disjoint family.

Definition I.2. A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is a *maximal almost disjoint family (of subsets of the natural numbers)* if it is infinite, consists of pairwise almost disjoint sets, and is not properly contained in another such family.

Existence of maximal almost disjoint families of subsets of the natural numbers is still unproblematic: we get such a family by applying Zorn's lemma to an infinite partition of $\mathbb{N}$ into infinite parts.

The following results on the cardinality of these maximal almost disjoint families are well known.

Theorem I.3. *Both Martin's axiom (*MA*) and the continuum hypothesis (*CH*) imply that all maximal almost disjoint families of subsets of the natural numbers are of cardinality continuum.*

Theorem I.4 ([22])**.** *In a model M of* $\mathsf{ZFC} + \mathsf{CH}$ *there exists a maximal almost disjoint family of subsets of the natural numbers that is still maximal almost disjoint in any Cohen extension of M.*

Theorem I.5. *There does not exist a countable maximal almost disjoint family of subsets of the natural numbers.*

With these theorems the following definition becomes reasonable.

Definition I.6. Let the cardinal $\mathfrak{a}$ be the least cardinality of a maximal almost disjoint family of subsets of the natural numbers.

The reason for the additional condition that the family be infinite in the definition of maximal almost disjoint family of subsets of the natural numbers is to keep the cardinal $\mathfrak{a}$ from being trivial. If we did not have the additional condition, then any partition of $\mathbb{N}$ into finitely many infinite pieces would be a maximal almost disjoint family.

The following theorem about $\mathfrak{a}$ is well known.

Theorem I.7 ([13])**.** *Suppose κ is a regular uncountable cardinal less than the continuum in a model of* ZFC*. Then there exists a forcing extension preserving cardinals* *and the cardinality of the continuum, in which there is a maximal almost disjoint family of subsets of the natural numbers of cardinality κ.*

The families $\mathcal{A}$ are subsets of $\mathcal{P}(\mathbb{N})$ which is a Polish space (a separable completely metrizable space). This is clear from the fact that $2^{\mathbb{N}}$ is, and $2^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ are homeomorphic as witnessed by the homeomorphism $S \in \mathcal{P}(\mathbb{N}) \mapsto \chi_S \in 2^{\mathbb{N}}$, where χ_S is the characteristic function of S. So we can apply descriptive set theoretic methods to these families in order to study their definability.

Question I.8. *How definable can a maximal almost disjoint family of subsets of the natural numbers be?*

The following two well known theorems answer this question.

Theorem I.9 ([26]). *A maximal almost disjoint family of subsets of the natural numbers cannot be analytic ($\mathbf{\Sigma}^1_1$).*

Theorem I.10 ([28]). *The axiom of constructibility implies that there exists a co-analytic ($\mathbf{\Pi}^1_1$) maximal almost disjoint family of subsets of the natural numbers.*

1.2 Families of Functions

We can now change the underlying set from $\mathcal{P}(\mathbb{N})$ to a different set and see what we get there. The first such set we consider is ${}^{\mathbb{N}}\mathbb{N}$, Baire space, the space of functions from $\mathbb{N}$ to $\mathbb{N}$. Note that this is also a Polish space. The usual definition of almost disjointness on this space is the following.

Definition I.11. We call $g_0, g_1 \in {}^{\mathbb{N}}\mathbb{N}$ *almost disjoint* or *eventually different* if the set $\{n \in \mathbb{N} : g_0(n) = g_1(n)\}$ is finite.

This is equivalent to the general notion of almost disjointness on this space if we consider functions from $\mathbb{N}$ to $\mathbb{N}$ to be subsets of $\mathbb{N} \times \mathbb{N}$. The corresponding notion of (maximal) almost disjoint family, called *maximal almost disjoint family of functions*, also requires no change from the general notion.

The analogues of Theorems I.3, I.4, and I.10 can be proved using similar methods. The following question is still open though (the analogue of Theorem I.9).

Question I.12. *Does there exist a Borel maximal almost disjoint family of functions?*

In working on this question Juris Steprāns introduced the following strengthening of the notion of maximal almost disjoint family of functions.

Definition I.13. (i). For two sets X, Y we say X is *almost contained* in Y, written $X \subseteq^* Y$, if $X \setminus Y$ is finite.

(ii). A function $f \in {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by a family $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ if there exist $g_0, \ldots, g_n \in \mathcal{A}$ such that $f \subseteq^* \bigcup_{i \leq n} g_i$ is finite.

(iii). A family $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by a family $\mathcal{A}$ if there exists an $f \in F$ that is finitely covered by $\mathcal{A}$ (note: only one function in the family needs to be finitely covered).

(iv). A family $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *strongly mad family* if it is an almost disjoint family of functions, and for every countable $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ that is not finitely covered by $\mathcal{A}$ there is a function $g \in \mathcal{A}$ such that for all $f \in F$ the intersection $f \cap g$ is infinite.

Note that any strongly mad family $\mathcal{A}$ is mad: if f is almost disjoint from all members of $\mathcal{A}$, then certainly $\{f\}$ is not finitely covered by $\mathcal{A}$. Since $\mathcal{A}$ is strongly mad this means there is $g \in \mathcal{A}$ such that $g \cap f$ is infinite contradicting that f is almost disjoint from all members of $\mathcal{A}$.

Steprāns showed the following theorem about this notion.

Theorem I.14 ([20])**.** *There does not exist an analytic* ($\mathbf{\Sigma}^1_1$) *strongly mad family.*

This result motivated us to study further the existence of strongly mad families. Note that with the additional covering condition imposed, the usual proof of existence using Zorn's lemma no longer works. In studying this existence question it quickly became apparent that all the results we obtained go through for a natural further strengthening.

Definition I.15. A family $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *very mad family* if it is an almost disjoint family and for every family $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ that is not finitely covered by $\mathcal{A}$ such that $|F| < |\mathcal{A}|$, there is a function $g \in \mathcal{A}$ such that for all $f \in F$ the intersection $f \cap g$ is infinite.

Note the following about these notions:

- Any very mad family is strongly mad (there do not exist countable mad families in Baire space).
- A strongly mad family of cardinality $\aleph_1$ is very mad.
- Under the continuum hypothesis the notions of very mad and strongly mad coincide.

We then proved the following theorems.

Theorem I.16 ([19])**.** *Martin's Axiom implies that very mad families exist and are of cardinality* $2^{\aleph_0}$.

Theorem I.17 ([19])**.** *Any model of* $\mathsf{ZFC} + \mathsf{CH}$ *contains a very mad family that is still very mad in any Cohen extension of that model.*

Theorem I.18 ([19])**.** *Suppose* κ *is a regular uncountable cardinal less than the continuum in a model of* ZFC*. Then there exists a forcing extension preserving* *cardinals and the cardinality of the continuum, in which there is a very mad family of cardinality κ.*

With these results the question of whether ZFC suffices to prove existence of very mad families is still open.

After this we showed with Yi Zhang a companion result to Steprāns' result (I.14).

Theorem I.19 ([20])**.** *The axiom of constructibility implies the existence of a coanalytic ($\mathbf{\Pi}^1_1$) very mad family.*

We noticed while working on very mad families that they are not as big as they seem; it is possible that there is very much space outside of them.

Definition I.20. Two very mad families $\mathcal{A}$, $\mathcal{B}$ are *orthogonal* if neither is finitely covered by the other.

We then proved the following theorem.

Theorem I.21 ([19])**.** *Martin's axiom implies the existence of a continuum size collection of very mad families that are pairwise orthogonal.*

And even more:

Theorem I.22. *The continuum hypothesis implies that for every mad family $\mathcal{A}$ there exists a very mad family $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal.*

1.3 Families and Groups of Permutations

We change the underlying space for the notion of almost disjointness again. Note that other than the change of space the notions here are identical to the general notion.

Definition I.23. (i). Let $\mathrm{Sym}(\mathbb{N}) \subseteq {}^{\mathbb{N}}\mathbb{N}$ denote the group of bijections from the natural numbers to the natural numbers with composition as the group operation.

(ii). A family of permutations $\mathcal{A} \subseteq \mathrm{Sym}(\mathbb{N})$ is *almost disjoint* if all distinct $f, g \in \mathcal{A}$ are almost disjoint. It is a *maximal almost disjoint family (of permutations)* if it is almost disjoint and not properly included in another such family.

Having an almost disjoint family being a subset of a group makes it very natural to impose the requirement that the family is a group, which gives us the following.

Definition I.24. A family $\mathcal{A} \subseteq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* if it is an almost disjoint family and it is a subgroup of $\mathrm{Sym}(\mathbb{N})$. It is a *maximal cofinitary group* if it is a cofinitary group not properly contained in another cofinitary group.

Here we should point out that this is not the usual definition of cofinitary group, although, of course, it is equivalent to it.

Definition I.25. (i). A permutation $g \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* if it has only finitely many fixed points.

(ii). We write $G \leq \mathrm{Sym}(\mathbb{N})$ if G is a subgroup of $\mathrm{Sym}(\mathbb{N})$.

The usual definition is given by the condition in the following theorem.

Theorem I.26. *A subgroup $G \leq \mathrm{Sym}(\mathbb{N})$ is a cofinitary group if all its non-identity members are cofinitary.*

Proof. For $f, g \in G$, n is a fixed point of $f^{-1}g$ iff $g(n) = f(n)$ iff $(n, g(n)) \in g \cap f$. □

The existence of both maximal almost disjoint families of permutations and maximal cofinitary groups follows, as it does for the general notions, from an argument using Zorn's lemma.

The following theorem was proved by Adeleke [1] and Truss [32].

Theorem I.27. *If $H \leq \mathrm{Sym}(\mathbb{N})$ is a maximal cofinitary group, then H is not countable.*

Also, P. Neumann proved the following result (see, e.g. [10, Proposition 10.4]).

Theorem I.28. *There exists a cofinitary group of cardinality $2^{\aleph_0}$.*

Thus, P. Cameron (in [10]) asked the following question.

Question I.29. *If the continuum hypothesis (CH) fails, is it possible that there exists a maximal cofinitary group H such that $|H| < 2^{\aleph_0}$?*

In [34], this question was answered by proving the following results.

Theorem I.30. *Martin's axiom implies that, if $H \leq \mathrm{Sym}(\mathbb{N})$ is a maximal cofinitary group, then H has cardinality $2^{\aleph_0}$.*

In the following theorem the notation $M^{\mathbb{P}} \models \varphi$ means that for any $\mathbb{P}$-generic set G over M the statement φ is true in $M[G]$.

Theorem I.31. *Let $M \models \mathsf{ZFC} + \neg\mathsf{CH}$. Let $\kappa \in M$ be a regular cardinal such that in M $\aleph_1 \leq \kappa < 2^{\aleph_0} = \lambda$. Then there exists a countable chain condition notion of forcing $\mathbb{P}$ such that the following statements hold in $M^{\mathbb{P}}$:*

(i). $2^{\aleph_0} = \lambda$;

(ii). there exists a maximal cofinitary group $H \leq \mathrm{Sym}(\mathbb{N})$ of cardinality κ.

The results corresponding to Theorem I.30 and Theorem I.31 for maximal almost disjoint families of permutations can be proved by similar methods. So both of the following two cardinal numbers are non-trivial.

Definition I.32. Define the cardinal $\mathfrak{a}_p$ to be the least cardinality of a maximal almost disjoint family of permutations, and similarly, $\mathfrak{a}_g$ to be the least cardinality of a maximal cofinitary group.

That these families can behave essentially differently from maximal almost disjoint families is witnessed by the following theorem.

Theorem I.33 ([9])**.** *It is consistent with* ZFC *that* $\mathfrak{a} < \mathfrak{a}_p = \mathfrak{a}_g$.

So the difference in which space the almost disjoint family is defined on can be used to get them to be of different cardinality (in [33] it was proved that $\mathfrak{a} < \mathfrak{a}_p$ is consistent, and in [16] that $\mathfrak{a} < \mathfrak{a}_g$ is consistent). The following question is however still open.

Question I.34 ([36])**.** *Is it consistent with* ZFC *that* $\mathfrak{a}_p$ *is distinct from* $\mathfrak{a}_g$*?*

That is, does the group structure influence the possible least cardinalities of almost disjoint families?

We next compare the cardinals $\mathfrak{a}_p$ and $\mathfrak{a}_g$ to some other well known cardinals which we define first.

Definition I.35. (i). Suppose that H is a group that is not finitely generated. Then H can be expressed as the union of a chain of proper subgroups. The *cofinality* of H, written $c(H)$, is the least λ such that H can be expressed as the union of a chain of λ proper subgroups.

(ii). $S \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *meager* if it is contained in the union of countably many nowhere dense sets. $\mathcal{M}$ is the collection of meager subsets of ${}^{\mathbb{N}}\mathbb{N}$.

(iii). non$(\mathcal{M})$, the *uniformity of meager sets*, is the size of the smallest non-meager set of reals.

(iv). $\mathcal{N}$ is the collection of Lebesgue null subsets of ${}^{\mathbb{N}}\mathbb{N}$.

(v). add($\mathcal{N}$), the *additivity of null sets*, is the least cardinality of a family $\mathcal{F} \subseteq \mathcal{N}$ such that $\cup\mathcal{F}$ is not a Lebesgue null set.

(vi). cof($\mathcal{N}$), the *cofinality of null sets*, is the least cardinality of a family $\mathcal{F} \subseteq \mathcal{N}$ such that for all $N \in \mathcal{N}$ there is $S \in \mathcal{F}$ such that $N \subseteq S$.

We first focus on the cofinality of the symmetric group. The following result was proved by H. D. Macpherson and P. Neumann in [24].

Theorem I.36. *If κ is an infinite cardinal, then $c(\mathrm{Sym}(\kappa)) > \kappa$.*

Upon learning of Theorem I.36, A. Mekler and S. Thomas independently pointed out the following easy observation (see, e.g. [30]).

Theorem I.37. *Suppose that $M \models \kappa^{\omega} = \kappa > \aleph_1$. Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ be the partial order of finite partial functions from κ to 2. Then $M^{\mathbb{P}} \models c(\mathrm{Sym}(\mathbb{N})) = \aleph_1 < 2^{\aleph_0} = \kappa$.*

Although MA implies $c(\mathrm{Sym}(\mathbb{N})) = 2^{\aleph_0}$ (see, e.g. [30]), some results indicate that $c(\mathrm{Sym}(\mathbb{N}))$ is rather small among the cardinal invariants. We give two examples:

(I) If $\mathfrak{d}$ is the dominating number (the minimum cardinality of a dominating family in ${}^{\mathbb{N}}\mathbb{N}$), then

Theorem I.38 ([30]). *$c(\mathrm{Sym}(\mathbb{N})) \leq \mathfrak{d}$.*

(II) A notion of forcing $\mathbb{P}$ is *Suslin* if and only if $\mathbb{P}$ is a $\mathbf{\Sigma^1_1}$ subset of $\mathbb{R}$ and both $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are $\mathbf{\Sigma^1_1}$ subsets of $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ denotes the reals. Then (see, e.g. [35]).

Theorem I.39. *Let $M \models \mathsf{ZFC} + \mathsf{GCH}$. Let $\mathbb{P}$ be a Suslin c.c.c. notion of forcing which adjoins reals, and let $\mathbb{Q}$ be the finite support iteration of $\mathbb{P}$ of length $\aleph_2$. Then $M^{\mathbb{Q}} \models c(\mathrm{Sym}(\mathbb{N})) = \aleph_1$.*

On the other hand, the following is a theorem of ZFC (see, e.g. [9]).

Theorem I.40. $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_p, \mathfrak{a}_g$.

As a corollary of Theorems I.39 and I.40, we know the following.

Corollary I.41. *It is consistent with* ZFC *that* $c(\mathrm{Sym}(\mathbb{N})) = \aleph_1 < \mathfrak{a}_p = \mathfrak{a}_g = 2^{\aleph_0} = \aleph_2$.

Proof. Iteratively add $\aleph_2$ random reals with finite support to a ground model $M \models \mathsf{ZFC} + \mathsf{GCH}$. (That $\mathrm{non}(\mathcal{M}) = 2^{\aleph_0}$ follows from the fact that any small set of reals already appears at some intermediate stage, and that in a random extension the set of ground model reals is meager, see [23, Theorem 3.20].) □

The obvious question left to answer is whether $\mathsf{ZFC} \vdash c(\mathrm{Sym}(\mathbb{N})) \leq \mathfrak{a}_p, \mathfrak{a}_g$. The following theorem shows that this does not hold.

Theorem I.42 ([21])**.** *It is consistent with* ZFC *that* $\mathfrak{a}_p = \mathfrak{a}_g < c(\mathrm{Sym}(\mathbb{N}))$.

Having shown that $\mathfrak{a}_g$ can be rather small, we turn to showing it can be very large.

The cardinals $\mathrm{add}(\mathcal{N})$, $\mathrm{non}(\mathcal{M})$, and $\mathrm{cof}(\mathcal{N})$, in Definition I.35, are some of the cardinals in Cichoń's diagram. For these it is known (see [2, Lemma 1.3.2]) that $\mathrm{add}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$ (in fact $\mathrm{add}(\mathcal{N})$ is the smallest cardinal in Cichoń's diagram, and $\mathrm{cof}(\mathcal{N})$ the largest). We'll construct, in a model of $\mathsf{ZFC} + \mathsf{CH}$, for any two cardinals $\lambda > \mu \geq \aleph_1$ a c.c.c. notion of forcing, using Shelah's recent technique of template forcing ([31]), such that in the forcing extension all cardinals in Cichoń's diagram are equal to μ, and $\mathfrak{a}_g = \lambda = 2^{\aleph_0}$ — that is,

Theorem I.43. *It is consistent with* ZFC *that* $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) < \mathfrak{a}_g = 2^{\aleph_0}$.

1.3.1 Definability

Just as for the other notions of almost disjointness, there is the question of how definable a maximal cofinitary group can be. Su Gao and Yi Zhang proved the following

Theorem I.44 ([12]). *The axiom of constructibility implies that there exists a maximal cofinitary group with a coanalytic generating set.*

We improved that to the following

Theorem I.45 ([17]). *The axiom of constructibility implies that there exists a coanalytic maximal cofinitary group.*

It is conjectured that there does not exist a Borel maximal cofinitary group. With the following lemma proved by Andreas Blass, this shows that the above theorem is conjecturally the best possible.

Lemma I.46 (Andreas Blass). *Any analytic maximal cofinitary group is Borel.*

Unfortunately the conjecture is quite far from being proved. For instance the following question is still open.

Question I.47. *Does there exist a closed maximal cofinitary group?*

There are some partial results though. Su Gao obtained the following theorem.

Theorem I.48. *There does not exist a compact maximal cofinitary group.*

We were able, with methods developed for one of the orbit results below, to improve this to the following. A set $S \subseteq \mathrm{Sym}(\mathbb{N})$ is a K_σ set if it is contained in a countable union of compact sets.

Theorem I.49. *There does not exist a K_σ maximal cofinitary group.*

Otmar Spinas proved the following theorem related to this question.

Theorem I.50. *There does not exist a locally compact maximal cofinitary group.*

1.3.2 Orbits and Isomorphism Types

Since $\mathrm{Sym}(\mathbb{N})$ has a natural action on $\mathbb{N}$, defined by $g \in \mathrm{Sym}(\mathbb{N})$ maps $n \in \mathbb{N}$ to $g(n)$, we can ask about the orbit structure of a maximal cofinitary group with this action. We proved the following results about this action.

Theorem I.51 ([18])**.** *A maximal cofinitary group has finitely many orbits.*

From the standard construction of maximal cofinitary groups using Martin's axiom or the continuum hypothesis, it is clear that non-transitive maximal cofinitary groups exist. The standard construction easily yields a maximal cofinitary group with any finite number of finite orbits. The following theorem required a new idea though.

Theorem I.52 ([18])**.** *Martin's axiom implies that for every $n, m \in \mathbb{N}$ with $m \geq 1$ there exists a maximal cofinitary group with n finite orbits and m infinite orbits.*

These theorems completely characterize the possible orbits of a maximal cofinitary group on $\mathbb{N}$ with respect to cardinality. The action of subgroups of $\mathrm{Sym}(\mathbb{N})$ on $\mathbb{N}$ generalizes to higher powers of $\mathbb{N}$.

Definition I.53. If $G \leq \mathrm{Sym}(\mathbb{N})$ then the *diagonal action* of G on $\mathbb{N}^k$ for some $k \in \mathbb{N}$ is given by $g(n_0, \ldots, n_{k-1}) = (g(n_0), \ldots, g(n_{k-1}))$.

The above theorems are then a first step towards answering the following question.

Question I.54. *Does a maximal cofinitary group have finitely many orbits under the diagonal action on $\mathbb{N}^k$ for any $k \in \mathbb{N}$?*

This question is related to the question on the complexity of maximal cofinitary groups in the following way (the statements and their proofs can be found in [15]). A group for which the answer to the question is affirmative is called oligomorphic. A closed subgroup of $\mathrm{Sym}(\mathbb{N})$ is the automorphism group of a countable first-order structure. Such a closed subgroup is oligomorphic if and only if it is the automorphism group of a countably categorical structure. So the answer to the question will give information on what sort of groups closed maximal cofinitary groups could be, of which type of structure they could be automorphism groups.

Any subgroup of $\mathrm{Sym}(\mathbb{N})$ is, obviously, of a certain abstract isomorphism type. The result of the usual constructions (both forcing and from Martin's axiom and the continuum hypothesis) is a group that has a generating set freely generating the group. In answering questions such as the one about the complexity of maximal cofinitary groups, only working with free maximal cofinitary groups is a restriction: it is possible that a maximal cofinitary group of least complexity is not a free group. Having good methods available to construct maximal cofinitary groups of different isomorphism types would therefore be beneficial for this work. So we work on the following question.

Question I.55. *What are the possible isomorphism types of maximal cofinitary groups?*

Related to this is the following question, which is also of independent interest.

Definition I.56. If G is an abstract group, it *has a cofinitary action* if there is an embedding $G \hookrightarrow \mathrm{Sym}(\mathbb{N})$ such that the image of the group under this embedding is a cofinitary group.

Question I.57. *Which abstract groups have cofinitary actions?*

Here we present our results which are some initial steps towards answering these questions.

Theorem I.58 ([18])**.** *Martin's axiom implies that there exists a maximal cofinitary group into which any countable group embeds.*

Any countable group has an obvious cofinitary action, its translation action on itself; so here we do not get new information about which groups can act cofinitarily. We do get a cofinitary group which is not free. We also proved that a group for which it is not a priori clear that it has a cofinitary action can consistently have a cofinitary action.

Theorem I.59 ([18])**.** *There exists a c.c.c. notion of forcing such that in the forcing extension the group* $\bigoplus_{\alpha\in\aleph_1}\mathbb{Z}_2$ *has a cofinitary action.*

The proof of this result yields also the following theorem.

Theorem I.60 ([18])**.** *In any model of Martin's axiom and the negation of the continuum hypothesis the group* $\bigoplus_{\alpha\in\aleph_1}\mathbb{Z}_2$ *has a cofinitary action.*

As Andreas Blass has observed, this group cannot have a maximal cofinitary action.

CHAPTER II

Very Mad Families

In this chapter we prove our results on very mad families. For convenience we will repeat the relevant definitions and give a short list of notational conventions used.

Definition II.1. ${}^{\mathbb{N}}\mathbb{N}$ is the space of functions from $\mathbb{N}$ to $\mathbb{N}$, called *Baire space*.

Often we will think of $f \in {}^{\mathbb{N}}\mathbb{N}$ as a subset of $\mathbb{N}\times\mathbb{N}$ (the subset $\{(n, f(n)) : n \in \mathbb{N}\}$). In the definitions below we will give statements equivalent to the versions where we don't use this idea; in the remainder of the chapter, however, we will not give these equivalents.

Definition II.2. (i). $g_0, g_1 \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint* (also called *eventually different*) if $g_0 \cap g_1$ is finite ($\{n \in \mathbb{N} : g_0(n) = g_1(n)\}$ is finite).

(ii). $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is an *almost disjoint family of functions* if any two distinct $g_0, g_1 \in \mathcal{A}$ are almost disjoint.

(iii). If $f \in {}^{\mathbb{N}}\mathbb{N}$ and $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$, then f is *finitely covered* by $\mathcal{A}$ if there are $g_0, \ldots, g_n \in \mathcal{A}$ such that $f \setminus \bigcup_{i\leq n} g_i$ is finite ($\{k \in \mathbb{N} : f(k) \notin \{g_i(k) : i \leq n\}$ is finite).

(iv). If $F, \mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ then F is *finitely covered* by $\mathcal{A}$ if there exists an $f \in F$ that is finitely covered by $\mathcal{A}$. (The notion of finitely covered is usually used in a negative context: if $F, \mathcal{A} \subseteq \mathcal{A}$ then F is not finitely covered by $\mathcal{A}$ if no element of F is finitely covered by $\mathcal{A}$.)

(v). $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *very mad family* if $\mathcal{A}$ is an almost disjoint family of functions, and for any $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that $|F| < |\mathcal{A}|$ and F is not finitely covered by $\mathcal{A}$, there exists $g \in \mathcal{A}$ such that for all $f \in F$, the set $f \cap g$ is infinite ($\{n \in \mathbb{N} : f(n) = g(n)\}$ is infinite).

Now we will give our guide to notation for this chapter.

$\mathcal{A}$ will be the very mad family under consideration (or parts thereof that we have already constructed). A will be a finite subset of $\mathcal{A}$. s will be a finite partial function $\mathbb{N} \rightharpoonup \mathbb{N}$. g or h (with sub- or superscripts) will be an element of $\mathcal{A}$. f will be any other element of ${}^{\mathbb{N}}\mathbb{N}$, often an element of F, which will be a family of functions not finitely covered by $\mathcal{A}$. $\bar{p} = \langle p_0, \ldots, p_n \rangle$ will be the sequence of length $\mathrm{lh}(\bar{p}) = n + 1$. We write π_i for the ith projection function; $\pi_i(\bar{p}) = p_i$.

2.1 Martin's Axiom

In this section we prove that MA implies the existence of very mad families.

We define for any subset $\mathcal{A}$ of Baire space a notion of forcing $\mathbb{P}_{\mathcal{A}}$, called *eventually different forcing*, see [2, page 366] and [27] (which will also be used in Sections 2.3 and 2.4). It consists of all conditions of the form $\langle s, A \rangle$ such that

- s is a finite partial function $\mathbb{N} \rightharpoonup \mathbb{N}$, and
- A is a finite subset of $\mathcal{A}$.

The ordering[1] $\langle s_2, A_2 \rangle \leq \langle s_1, A_1 \rangle$ is defined by

$$s_1 \subseteq s_2 \ \wedge \ A_1 \subseteq A_2 \ \wedge \ \forall g \in A_1 \, [g \cap s_2 \subseteq s_1].$$

[1] $q \leq p$ means that q is an extension of p.

This notion of forcing is σ-centered (a strengthening of c.c.c., for the definition see [2, p. 23]) since there are only countably many choices for s in a condition and any two conditions with identical first coordinate are compatible. This shows that in fact for the following lemma and theorem the hypothesis $\mathfrak{p} = 2^{\aleph_0}$ (here $\mathfrak{p}$ is pseudointersection number) suffices, since this equality implies MA(σ-centered) ([4] has a nice presentation of forcing axioms where this implication is proved; this result originally appeared in [3]).

Since CH implies MA, the results below are also true under CH. Of course in the case of CH you can do essentially the same construction without mentioning a notion of forcing.

Lemma II.3 (MA)**.** *Assume that $\mathcal{A}$ is an almost disjoint family of functions with $|\mathcal{A}| < 2^{\aleph_0}$ and that F is a family not finitely covered by $\mathcal{A}$ with $|F| < 2^{\aleph_0}$. Then there exists a function $g \notin \mathcal{A}$ such that:*

(i). $\mathcal{A} \cup \{g\}$ is an almost disjoint family of functions, and

(ii). for all $f \in F$, the set $f \cap g$ is infinite.

Proof. For each $f \in F$, $h \in \mathcal{A}$ and $n \in \mathbb{N}$ let

- $C_h := \{\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}} : h \in A\}$;
- $D_n := \{\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}} : n \in \mathrm{dom}(s)\}$;
- $E_{f,n} := \{\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}} : \exists m \geq n\ f(m) = s(m)\}$.

All these sets are dense in $\mathbb{P}_{\mathcal{A}}$: Let $\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}}$. For C_h we have $\langle s, A\rangle \geq \langle s, A \cup \{h\}\rangle \in C_h$. For D_n we need to find an extension of s such that n is in its domain (if this is not already the case). Since A is finite, there exists an m such that $m \neq g_0(n)$ for all $g_0 \in A$ and we take $s \cup \{(n, m)\}$. For $E_{f,n}$ note that since f

is not finitely covered by $\mathcal{A}$, there is an $m > n$ such that $f(m) \notin \{g_0(m) : g_0 \in A\}$. Then $\langle s, A\rangle \geq \langle s \cup \{(m, f(m))\}, A\rangle \in E_{f,n}$.

The family

$$\mathcal{D} = \{C_h : h \in \mathcal{A}\} \cup \{D_n : n \in \mathbb{N}\} \cup \{E_{f,n} : f \in F, n \in \mathbb{N}\}$$

has cardinality less than $2^{\aleph_0}$. Therefore by MA there is a filter G meeting all of the sets in $\mathcal{D}$. Then $g := \bigcup\{s : \exists A \subseteq \mathcal{A}\ [\langle s, A\rangle \in G]\}$ is the desired function ($\{D_n : n \in \mathbb{N}\}$ ensure that it is a function with domain all of $\mathbb{N}$, $\{C_h : h \in \mathcal{A}\}$ together with the definition of the order ensure g is almost disjoint from all members of $\mathcal{A}$, and for each $f \in F$, $\{E_{f,n} : n \in \mathbb{N}\}$ ensure that $f \cap g$ is infinite). □

Theorem II.4 (MA)**.** *There exists a very mad family, and any such family is of cardinality* $2^{\aleph_0}$.

Proof. We shall construct functions g_α such that $\{g_\alpha : \alpha < 2^{\aleph_0}\}$ is a very mad family of size $2^{\aleph_0}$. Let $\{f_\alpha : \alpha < 2^{\aleph_0}\}$ be an enumeration of ${}^{\mathbb{N}}\mathbb{N}$. At stage β we do the following.

Let F be the maximal subset of $\{f_\alpha : \alpha < \beta\}$ that is not finitely covered by $A_\beta := \{g_\alpha : \alpha < \beta\}$ (F is the set of functions in $\{f_\alpha : \alpha < \beta\}$ that are not finitely covered by A_β). By Lemma II.3 there exists g_β such that $A_\beta \cup \{g_\beta\}$ is an almost disjoint family of functions, and for any $f \in F$ the set $f \cap g_\beta$ is infinite (if F is empty, we just get a new function almost disjoint from all of A_α).

Now let $\mathcal{A} = \{g_\beta : \beta < 2^{\aleph_0}\}$; we claim that $\mathcal{A}$ is a very mad family. Let F be not finitely covered by $\mathcal{A}$ and such that $|F| < |\mathcal{A}| = 2^{\aleph_0}$. Since F is of cardinality less than $2^{\aleph_0}$, we have $F \subseteq \{f_\alpha : \alpha < \beta\}$ for some $\beta < 2^{\aleph_0}$ (MA implies that $2^{\aleph_0}$ is regular), and then g_β will meet all members of F infinitely often.

The second clause follows immediately from Lemma II.3. □

2.2 In the Cohen Model

We prove that in any model of the continuum hypothesis there is a very mad family that survives Cohen forcing. For this we need the following lemma, which is from [22, Lemma 2.2, p. 256].

We let $\mathrm{Fn}(I, 2)$ denote the set of finite partial functions $I \rightharpoonup 2$ ordered by reverse inclusion.

Lemma II.5. *Let M be a model of ZFC, $I, S \in M$, G be $\mathrm{Fn}(I, 2)$-generic over M, and $X \subseteq S$ with $X \in M[G]$. Then $X \in M[G \cap \mathrm{Fn}(I_0, 2)]$ for some $I_0 \subseteq I$ such that $I_0 \in M$ and $(|I_0| \leq |S|)^{M}$.* □

Theorem II.6. *Let M be a model of $\mathsf{ZFC} + \mathsf{CH}$. Then there is a very mad family $\mathcal{A}$ in M such that for any $I \in M$ and $\mathrm{Fn}(I, 2)$-generic set G, $M[G] \models$ "$\check{\mathcal{A}}$ is a very mad family of size $\aleph_1$".*

Proof. We construct a very mad family that survives forcing with $\mathrm{Fn}(\mathbb{N}, 2)$, and then show that this family survives forcing with $\mathrm{Fn}(I, 2)$ for any I. Note that since the continuum hypothesis is true in M, strongly and very mad families in M are the same thing, and that the very mad family in M will be of size $\aleph_1$, so it only has to contain functions capturing any countable collection in the extension.

Since in M

$$\begin{aligned} &|\,\mathrm{Fn}(\mathbb{N}, 2) \times \{\tau : \tau \text{ is a nice name for a subset of } (\mathbb{N} \times \mathbb{N})\check{}\ \}| \\ &\leq \aleph_0 \times \left((\#\text{ anti-chains in } \mathrm{Fn}(\mathbb{N}, 2)) \times |\{0,1\}^{\aleph_0}|\right)^{|\mathbb{N}\times\mathbb{N}|} \\ &\leq \aleph_0 \times (2^{\aleph_0} \times 2^{\aleph_0})^{\aleph_0} \overset{\mathsf{CH}}{=} \aleph_1, \end{aligned}$$

we can enumerate $\mathrm{Fn}(\mathbb{N}, 2) \times \{\tau : \tau \text{ is a nice name for a subset of } (\mathbb{N} \times \mathbb{N})\check{}\ \}$ as $\langle (p_i, \tau_i) : i < \omega_1 \rangle$.

We construct $\mathcal{A} = \{g_\alpha : \alpha < \omega_1\}$, the very mad family, recursively.

Assume that all g_α for $\alpha < \beta$ have been defined. We need g_β to satisfy:

G1. for all $\alpha < \beta$, the functions g_α and g_β are almost disjoint, and

G2. if $F_\beta := \{\tau_\alpha : \alpha < \beta, p_\beta \Vdash$ "τ_α is a total function and τ_α is not finitely covered by $\{\check{g}_\gamma : \gamma < \check{\beta}\}$"$\}$, then $(\forall \tau_\alpha \in F_\beta)\ p_\beta \Vdash$ "$|\tau_\alpha \cap \check{g}_\beta| = \check{\omega}$".

$p_\beta \Vdash$ "$|\tau_\alpha \cap \check{g}_\beta| = \check{\omega}$" is equivalent to

$$(\forall n)(\forall q \leq p_\beta)(\exists r \leq q)(\exists m \geq n)\ r \Vdash \text{"}\tau_\alpha(\check{m}) = \check{g}_\beta(\check{m})\text{"}.$$

Enumerate $\mathbb{N} \times \{q : q \leq p_\beta\}$ as $\langle (n_i, q_i) : i < \omega\rangle$, $\langle g_\alpha : \alpha < \beta\rangle$ as $\langle g'_i : i < \omega\rangle$ and F_β as $\langle \tau'_i : i < \omega\rangle$.

Recursively define g_β. Before stage s we have g_β defined on $\{0, \ldots, n_s\}$. At stage s we want to define g_β on $\{n_s + 1, \ldots, n_{s+1}\}$ for some n_{s+1} so that G1 and G2 will eventually be satisfied.

The requirements at this stage will be:

L1. $\forall n \in \{n_s + 1, \ldots, n_{s+1}\}\ (g_\beta(n) \notin \{g'_0(n), \ldots, g'_s(n)\})$, and

L2. there are $n_{s,0}, \ldots, n_{s,s} > n_s$ all distinct and $r_{s,0}, \ldots, r_{s,s} \leq q_s$ such that for all i from 0 to s we have $r_{s,i} \Vdash$ "$\tau'_i(\check{n}_{s,i}) = \check{g}_\beta(\check{n}_{s,i})$".

Requirement L1 ensures that g_β will satisfy G1, and requirement L2 ensures that g_β will satisfy G2.

Since $p_\beta \Vdash$ "τ'_i is a total function not finitely covered by $\check{g}'_0, \ldots, \check{g}'_s$", for every $m > n_s$ there is an $n_{s,i} > m$ and $r'_{s,i} \leq q_s$ such that $r'_{s,i} \Vdash$ "$\tau'_i(\check{n}_{s,i}) \notin \{\check{g}'_i(\check{n}_{s,i}) : 0 \leq i \leq s\}$". Then below $r'_{s,i}$ there is a condition $r_{s,i}$ that decides the value of $\tau'_i(\check{n}_{s,i})$.

We use this observation repeatedly to find $n_{s,0} < \cdots < n_{s,s}$ all larger than n_s, and define $g_\beta(n_{s,i})$ to be the number that $\tau'_i(\check{n}_{s,i})$ is forced to be by $r'_{s,i}$. Then we set

$n_{s+1} = n_{s,s}$ and set $g_\beta(n)$, for $n < n_{s+1}$ such that $g_\beta(n)$ is not defined yet, to be any number not in $\{g'_i(n) : 0 \leq i \leq s\}$. This completes the construction.

We show that $\mathcal{A} = \{g_\beta : \beta < \omega_1\}$ is a very mad family in the forcing extension $M[G]$, for any G that is $\mathrm{Fn}(\mathbb{N}, 2)$-generic over M. First note that $\mathcal{A}$ is an almost disjoint family in $M[G]$, since it is almost disjoint in M: the functions are almost disjoint by the first requirement. To see that it is very mad, let F be a countable family of functions, all of which are not finitely covered by $\mathcal{A}$. Then in $M[G]$ we have $F \subseteq \{\tau_\beta[G] : \beta < \omega_1\}$; therefore there exists an α (by the countability of F) such that $F \subseteq \{\tau_\beta[G] : \beta < \alpha\}$, and this is forced by some $p \in G$. Then at some point β in the construction (after stage α) when we have a p_β equal to p we will correctly deal with a superset of F (and therefore with F), by the second requirement.

It remains to show that for any I the family $\mathcal{A}$ is very mad in any forcing extension $M[G]$ with G $\mathrm{Fn}(I, 2)$-generic over M. Suppose that in some forcing extension by $\mathrm{Fn}(I, 2)$ the family $\mathcal{A}$ defined above is no longer very mad. There is then a countable family F of functions not finitely covered by $\mathcal{A}$ for which there does not exist $g \in \mathcal{A}$ such that for all $f \in F$ the set $f \cap g$ infinite. Code the family in a single real. Then there is a countable I_0 such that this real, and therefore the family F, are in the extension of M by $\mathrm{Fn}(I_0, 2) \cap G$ (by Lemma II.5). Since $\mathrm{Fn}(I_0, 2) \cong \mathrm{Fn}(\mathbb{N}, 2)$ the above argument for $\mathrm{Fn}(\mathbb{N}, 2)$ shows that there is a function $g \in \mathcal{A}$ such that for all $f \in F$ the set $f \cap g$ is infinite. This contradicts the existence of such a family F. (Note that this also shows $\mathcal{A}$ is very mad in M, by taking $I = \emptyset$.) $\square$

2.3 Orthogonality Results

In this section we'll prove the existence, under Martin's axiom, of many orthogonal very mad families (see Definition I.20). This intuitively shows that there can be much "room" outside of a very mad family. We also show that under the continuum hypothesis, for any very mad family there exists a very mad family orthogonal to it.

Let $\langle \mathcal{A}_\alpha : \alpha < \beta \rangle$ be subsets of Baire space. Define the notion of forcing $\mathbb{Q}_{\langle \mathcal{A}_\alpha : \alpha < \beta \rangle}$, to consist of all conditions $\bar{p}$ that are finite partial functions with domain contained in β and $\bar{p}(\alpha) \in \mathbb{P}_{\mathcal{A}_\alpha}$ (from Section 2.1). Define $\bar{q} \leq \bar{p}$ iff $\mathrm{dom}(\bar{p}) \subseteq \mathrm{dom}(\bar{q})$ and $\forall \alpha \in \mathrm{dom}(\bar{p})\ [\bar{q}(\alpha) \leq_{\mathbb{P}_{\mathcal{A}_\alpha}} \bar{p}(\alpha)]$.

Lemma II.7 (MA). *Let $\mathcal{A}_\alpha$ ($\alpha < \beta < 2^{\aleph_0}$) be almost disjoint families that are pairwise orthogonal, and let F_α ($\alpha < \beta$) be families of functions such that $|F_\alpha| < 2^{\aleph_0}$ and F_α is not finitely covered by $\mathcal{A}_\alpha$. Then there exist functions $g_\alpha \notin \mathcal{A}_\alpha$ ($\alpha < \beta$) such that*

(i). for all $\alpha < \beta$, the family $\mathcal{A}_\alpha \cup \{g_\alpha\}$ is an almost disjoint family,

(ii). for all $\alpha < \beta$ and $f \in F_\alpha$, the set $f \cap g_\alpha$ is infinite, and

(iii). for all $\alpha_1, \alpha_2 < \beta$, the families $A_{\alpha_1} \cup \{g_{\alpha_1}\}$ and $A_{\alpha_2} \cup \{g_{\alpha_2}\}$ are orthogonal.

Proof. In addition to the dense sets used in the proof of Lemma II.3 for each coordinate $\alpha < \beta$, define for all $\alpha_1 \neq \alpha_2 < \beta, n \in \mathbb{N}$ and $a \in [\mathcal{A}_{\alpha_2}]^{<\omega}$ the sets (remember π_i is the ith projection function, see page 18)

$$A_{\alpha_1,\alpha_2,a,n} := \{\bar{p} : a \subseteq \pi_1(\bar{p}(\alpha_2)) \ \wedge \exists m > n \big[m \in \mathrm{dom}(\pi_0(\bar{p}(\alpha_1))) \ \wedge$$
$$m \in \mathrm{dom}(\pi_0(\bar{p}(\alpha_2))) \ \wedge \pi_0(\bar{p}(\alpha_1))(m) \notin \{f(m) : f \in a\} \cup \{\pi_0(\bar{p}(\alpha_2))(m)\} \big] \}.$$

These sets are easily seen to be dense, and combined with the dense sets from Lemma II.3 there are still fewer than continuum many. So MA implies there is a filter G meeting them all. Then set $g_\alpha := \cup\{\bar{p}(\alpha) : \bar{p} \in G\}$.

The dense sets from Lemma II.3 guarantee the first two items of the theorem, and the new dense sets $A_{\alpha_1,\alpha_2,a,n}$ ensure that the resulting families are still orthogonal: the filter intersecting all dense sets $A_{\alpha_1,\alpha_2,a,n}$ for $n \in \mathbb{N}$ ensures that g_{α_1} is not finitely covered by $a \cup \{g_{\alpha_2}\}$. □

Theorem II.8 (MA)**.** *There exist very mad families $\mathcal{A}_\alpha$ ($\alpha < 2^{\aleph_0}$) such that for all $\alpha_1 \neq \alpha_2 < 2^{\aleph_0}$ the families $\mathcal{A}_{\alpha_1}$ and $\mathcal{A}_{\alpha_2}$ are orthogonal.*

Proof. We will construct the families $\mathcal{A}_\alpha$ recursively as $\mathcal{A}_\alpha := \bigcup_{\gamma<2^{\aleph_0}} \mathcal{A}_{\alpha,\gamma}$.

We start with $\mathcal{A}_{\alpha,0} := \emptyset$ for all $\alpha < 2^{\aleph_0}$, and we take unions at limit stages. Then at step β we apply Lemma II.7 with $\mathbb{Q}_{\langle \mathcal{A}_{\alpha,\beta}:\alpha<\beta\rangle}$ to obtain $\langle g_\alpha : \alpha < \beta\rangle$ and set $\mathcal{A}_{\alpha,\beta+1} := \mathcal{A}_{\alpha,\beta} \cup \{g_\alpha\}$ for $\alpha < \beta$, and $\mathcal{A}_{\alpha,\beta+1} := \mathcal{A}_{\alpha,\beta} = \emptyset$ for $\beta \leq \alpha < 2^{\aleph_0}$. □

Now we move towards proving the following theorem whose proof will take up the remainder of this section.

Theorem II.9 (CH)**.** *For every very mad family $\mathcal{A}$ there exists a very mad family $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal.*

We will consistently use the convention that barred functions are to be thought of as related to $\mathcal{B}$ where the unbarred versions relate to $\mathcal{A}$. First we work out the ideas needed to be able to construct a family $\mathcal{B}$ not finitely covering $\mathcal{A}$.

Lemma II.10. *Let $\bar{g}_0, \ldots, \bar{g}_n \in {}^{\mathbb{N}}\mathbb{N}$, $g_0, \ldots g_n \in \mathcal{A}$ and $W \in [\mathbb{N}]^{\aleph_0}$ such that for all $i \leq n$, $\mathrm{dom}(\bar{g}_i \cap g_i) \supseteq W$. Then*

$$g \in \mathcal{A} \ \wedge \ g \restriction W \subseteq^* \bigcup_{i \leq n} \bar{g}_i \ \Rightarrow \ \text{there is } i \leq n \text{ such that } g = g_i$$

Proof. The partial function $g \restriction W$ agrees on infinitely many inputs with some $\bar{g}_i \restriction W$. This means that $g \restriction W$ agrees on infinitely many inputs with g_i, which in turn gives $g = g_i$ by almost disjointness of $\mathcal{A}$. □

From this we immediately get:

Corollary II.11. *Under the same hypothesis, if $g \in \mathcal{A}$ is not one of the functions $g_0, \ldots, g_n$ then $g \restriction W \setminus \bigcup_{i \leq n} \bar{g}_i$ is infinite.*

Definition II.12. For a family $\mathcal{B} \subseteq {}^{\mathbb{N}}\mathbb{N}$, a function $H : [\mathcal{B}]^{<\aleph_0} \to ([\mathbb{N}]^{\aleph_0})^2 \times ([\mathcal{A}]^{<\aleph_0})^2$ is *good for $\mathcal{B}$* if for all $\{\bar{g}_0, \ldots, \bar{g}_n\} \in [\mathcal{B}]^{<\aleph_0}$ we have that $H(\{\bar{g}_0, \ldots, \bar{g}_n\}) = \langle W_0, W_1, \{g_0^0, \ldots, g_n^0\}, \{g_0^1, \ldots, g_n^1\}\rangle$ such that for all $i, j \leq n$ and $k, l \in \{0, 1\}$,

$$i \neq j \ \vee \ k \neq l \ \Rightarrow \ g_i^l \neq g_j^k,$$

and for all $i \leq n$,

$$\mathrm{dom}(\bar{g}_i \cap g_i^0) \supseteq W_0 \ \wedge \ \mathrm{dom}(\bar{g}_i \cap g_i^1) \supseteq W_1.$$

Lemma II.13. *If $\mathcal{B}$ is such that an H good for $\mathcal{B}$ exists, then $\mathcal{B}$ does not finitely cover $\mathcal{A}$.*

Proof. Suppose that $g \in \mathcal{A}$ is such that there are $\bar{g}_0, \ldots \bar{g}_n \in \mathcal{B}$ such that $g \subseteq^* \bigcup_{i \leq n} \bar{g}_i$. Let $H(\{\bar{g}_0, \ldots, \bar{g}_n\}) = \langle W_0, W_1, \{g_0^0, \ldots, g_n^0\}, \{g_0^1, \ldots, g_n^1\}\rangle$. Then as $g \restriction W_0 \subseteq^* \bigcup_{i \leq n} \bar{g}_i$, we have $g = g_i^0$ for some $i \leq n$ (by Lemma II.10), but also $g \restriction W_1 \subseteq^* \bigcup_{i \leq n} \bar{g}_i$, which gives $g = g_j^1$ for some $j \leq n$ (Lemma II.10 again). So $g_i^0 = g_j^1$, contradicting H being good for $\bar{A}$. $\square$

Now that we know how to take care of not finitely covering $\mathcal{A}$ (it is enough to ensure existence of a function good for $\mathcal{B}$), we look into being not finitely covered by $\mathcal{A}$.

Lemma II.14. *If $\bar{g} \in {}^{\mathbb{N}}\mathbb{N}$ is such that there exist $\langle g_n : n \in \mathbb{N}\rangle$ with all $g_n \in \mathcal{A}$ different and $\bar{g}$ agrees with each g_n on infinitely many inputs, then $\bar{g}$ is not finitely covered by $\mathcal{A}$.*

Proof. Suppose $\bar{g} \subseteq^* \bigcup_{i \leq n} g'_i$ for $g'_i \in \mathcal{A}$. Then every g_n agrees infinitely often with some g'_j. As there are infinitely many g_n, some g'_j infinitely often agrees with two different g_n. This contradicts the almost disjointness of $\mathcal{A}$. □

Now we are ready for the construction. With CH we can enumerate ${}^{\mathbb{N}}\mathbb{N}$ by $\langle f_\alpha : \alpha < \omega_1 \rangle$ and also using $\omega_1 \cong \mathbb{N} \times \omega_1$ we can enumerate $\mathcal{A} = \langle g_{n,\alpha} : n \in \mathbb{N} \wedge \alpha < \omega_1 \rangle$ without repetitions.

We will construct $\mathcal{B}$ and H by recursion as $\langle \bar{g}_\alpha : \alpha < \omega_1 \rangle$ and $\bigcup_{\alpha < \omega_1} H_\alpha$.

We call H_α good if it is good for $\langle \bar{g}_\beta : \beta < \alpha \rangle$ with the following added requirements:

H1. if $S \in [\{\bar{g}_\beta : \beta < \alpha\}]^{<\aleph_0}$, then $\pi_2(H(S)) \cup \pi_3(H(S)) \subseteq \{g_{n,\beta} : n \in \mathbb{N} \ \wedge \ \beta < \alpha\}$;

H2. if $S_0, S_1 \in [\{\bar{g}_\beta : \beta < \alpha\}]^{<\aleph_0}$ and $S_0 \subseteq S_1$, then $\pi_0(H(S_1)) \subseteq \pi_0(H(S_0))$ and $\pi_1(H(S_1)) \subseteq \pi_1(H(S_0))$.

Note that if λ is a limit ordinal and if all H_α for $\alpha < \lambda$ are good, then $\bigcup_{\alpha<\lambda} H_\alpha$ is good for $\langle \bar{g}_\alpha : \alpha < \lambda \rangle$. For limit ordinals λ we take $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$.

Let $\alpha < \omega_1$ and assume that $\langle \bar{g}_\beta : \beta < \alpha \rangle$ is an almost disjoint family orthogonal to $\mathcal{A}$ and H_α is good. It suffices to construct $\bar{g}_\alpha$ and $H_{\alpha+1}$ such that

G1. $\bar{g}_\alpha$ is almost disjoint from $\langle \bar{g}_\beta : \beta < \alpha \rangle$,

G2. $H_{\alpha+1}$ is good,

G3. $\bar{g}_\alpha$ agrees on infinitely many inputs with infinitely many members of $\mathcal{A}$, and

G4. all f_β, for $\beta < \alpha$, that are not finitely covered by $\langle \bar{g}_\beta : \beta < \alpha \rangle$ agree with $\bar{g}_\alpha$ on infinitely many inputs.

Now in order to take care of this we do some reenumeration: enumerate $\{\bar{g}_\beta : \beta < \alpha\}$ by $\langle \bar{g}'_n : n \in \mathbb{N} \rangle$, enumerate the set $\{f_\beta : \beta < \alpha \wedge f_\beta$ is not finitely covered by $\{\bar{g}'_n :$

$n \in \mathbb{N}\}\}$ by $\langle f'_n : n \in \mathbb{N}\rangle$ and enumerate the set $[\{\bar{g}'_n : n \in \mathbb{N}\}]^{<\aleph_0}$ by $\langle S_n : n \in \mathbb{N}\rangle$. Then write $W_{n,0}$ for $\pi_0(H_\alpha(S_n))$ and $W_{n,1}$ for $\pi_1(H_\alpha(S_n))$.

We will construct $\bar{g}_\alpha$ recursively. After step $s \in \mathbb{N}$ we have $\bar{g}_\alpha$ defined on an initial segment $[0, n_s]$. During step $s+1$ we satisfy the following requirements.

L1. $\bar{g}_\alpha(n) \notin \{\bar{g}'_i(n) : i \leq s\}$,

L2. For all $n \leq s$ there exist $k \in W_{n,0}$ and $l \in W_{n,1}$ such that $\bar{g}_\alpha(k) = g_{0,\alpha}(k)$ and $\bar{g}_\alpha(l) = g_{1,\alpha}$,

L3. For all $n \leq s$ there exists a k where $\bar{g}_\alpha$ was not yet defined before this step, such that $\bar{g}_\alpha(k) = g_{n,\alpha}(k)$, and

L4. For all $n \leq s$ there exists a k where $\bar{g}_\alpha$ was not yet defined before this step, such that $\bar{g}_\alpha(k) = f_n(k)$.

A detailed construction taking care of L1–L4 is the following:

- Since H_α is good, H1, H2 and Corollary II.11 give that $g_{0,\alpha} \restriction W_{n,0} \setminus (\bigcup S_n) \cup (\bigcup_{i \leq s} \bar{g}'_s)$ is infinite: H2 gives $\pi_0(H_\alpha(S_n \cup \{\bar{g}_i : i \leq s\})) \subseteq W_{n,0}$, H1 gives that $g_{0,\alpha}$ is not in $H_\alpha(S_n \cup \{\bar{g}_i : i \leq s\})$ so Corollary II.11 applies. Similarly $g_{1,\alpha} \restriction W_{n,1} \setminus (\bigcup S_n) \cup (\bigcup_{i \leq s} \bar{g}'_s)$ is infinite.

  So we can choose $w^0_{s,i} \in W_{i,0}$, $w^1_{s,i} \in W_{i,1}$ for $i \leq s$, such that $n_s < w^0_{s,0} < \cdots < w^0_{s,s} < w^1_{s,0} < \cdots < w^1_{s,s}$ with $g_{0,\alpha}(w^0_{s,i}) \notin \{\bar{g}'_i(w^0_{s,i}) : i \leq s\}$ and $g_{0,\alpha}(w^1_{s,i}) \notin \{\bar{g}'_i(w^1_{s,i}) : i \leq s\}$.

  Define $\bar{g}_\alpha(w^0_{s,i}) := g_{0,\alpha}(w^0_{s,i})$ and $\bar{g}_\alpha(w^1_{s,i}) := g_{0,\alpha}(w^1_{s,i})$. This takes care of L2 while respecting L1.

- As $g_{i,\alpha}$ is not finitely covered by $\{\bar{g}'_n : n \in \mathbb{N}\}$ (Lemma II.13), for every $i \leq s$ we can find $n_{s,i}$ such that $w^1_{s,s} < n_{s,0} < \cdots < n_{s,s}$ and $g_{i,\alpha}(n_{s,i}) \notin \{\bar{g}'_n(n_s, i) : n \leq s\}$.

Define $\bar{g}_\alpha(n_{s,i}) := g_{i,\alpha}(n_{s,i})$. This takes care of L3 while respecting L1.

- As f'_i is not finitely covered by $\{\bar{g}'_n : n \in \mathbb{N}\}$ for every $i \leq s$, we can find $m_{s,i}$ such that $n_{s,s} < m_{s,0} < \cdots < m_{s,s}$ and $f'_i(m_{s,i}) \notin \{\bar{g}_j(m_{s,i}) : j \leq s\}$.

  Define $\bar{g}_\alpha(m_{s,i}) := f'_i(m_{s,i})$. This takes care of L4 while respecting L1.

- Now for any n such that $n_s < n < m_{s,s}$ where $\bar{g}_\alpha$ is not defined, define $\bar{g}_\alpha(n)$ to be the least number not in $\{\bar{g}'_i(n) : i \leq s\}$, and set $n_{s+1} = m_{s,s}$.

This construction satisfies the requirements L1–L4. These in turn imply the requirements G1–G4; this is immediate for all but G2, where we still have to define

$$H_{\alpha+1} : [\{\bar{g}_\beta : \beta \leq \alpha\}]^{<\aleph_0} \to ([\mathbb{N}]^{\aleph_0})^2 \times ([\{g_{n,\beta} : n \in \mathbb{N}, \beta \leq \alpha\}^{<\aleph_0}])^2.$$

If $S \in [\{\bar{g}_\beta : \beta \leq \alpha\}]^{<\aleph_0}$ then $S \subseteq \{\bar{g}_\beta : \beta < \alpha\}$ or $S = S' \cup \{\bar{g}_\alpha\}$ with $S' \subseteq \{\bar{g}_\beta : \beta < \alpha\}$.

In the first case, set $H_{\alpha+1}(S) := H_\alpha(S)$; otherwise $S' = S_n$ for some $n \in \mathbb{N}$, and if $H_\alpha(S_n) = \langle W_0, W_1, \{g^0_0, \ldots, g^0_n\}, \{g^1_0, \ldots, g^1_n\}\rangle$ set $H_{\alpha+1}(S) := \langle \{w^0_{s,n} : n \leq s\}, \{w^1_{s,n} : n \leq s\}, \{g^0_0, \ldots, g^0_n, g_{0,\alpha}\}, \{g^1_0, \ldots, g^1_n, g_{1,\alpha}\}\rangle$.

This $H_{\alpha+1}$ is good as can be easily checked, completing the proof.

2.4 A Very Mad Family Not of Size Continuum

In this section we show that consistently there exist very mad families of any uncountable cardinality less than or equal to the continuum. The forcing is based on the proof of the similar result for maximal cofinitary groups by Yi Zhang in [34].

Theorem II.15. *Let M be a model of* ZFC *and assume that, in M, κ is a regular cardinal such that $\aleph_1 \leq \kappa < 2^{\aleph_0} = \lambda$. Then there exists a c.c.c. forcing $\mathbb{P}$ such that $M^{\mathbb{P}}$ satisfies* *(i).* $2^{\aleph_0} = \lambda$, *and*

(ii). there exists a very mad family $\mathcal{A}$ *of cardinality* κ.

In the proof we will use the c.c.c. poset $\mathbb{P}_{\mathcal{A}}$ from section 2.1. First we prove its main property (which is basically Lemma II.3 rephrased in the language of forcing).

Lemma II.16. *If* N *is a model of* ZFC, $f \in N \cap {}^{\mathbb{N}}\mathbb{N}$, $\mathcal{A} \subset {}^{\mathbb{N}}\mathbb{N}$, $\mathcal{A} \in N$ *and* f *is not finitely covered by* $\mathcal{A}$, *then the generic function* $g \notin N$ *obtained from forcing with* $\mathbb{P}_{\mathcal{A}}$ *over* N *satisfies*

(i). $\mathcal{A} \cup \{g\}$ *is an almost disjoint family;*

(ii). $f \cap g$ *is infinite.*

Proof. We use the notation from the proof of Lemma II.3. g is a total function since for every $n \in \mathbb{N}$, $D_n = \{\langle s, H\rangle \in \mathbb{P}_{\mathcal{A}} : n \in \mathrm{dom}(s)\}$ is dense and in N. $\mathcal{A} \cup \{g\}$ is almost disjoint since for every $h \in \mathcal{A}$ the set $C_h = \{\langle s, H\rangle \in \mathbb{P}_{\mathcal{A}} : h \in H\}$ is dense and in N. $f \cap g$ is infinite as for every $n \in \mathbb{N}$ the set $E_{f,n} = \{\langle s, H\rangle \in \mathbb{P}_{\mathcal{A}} : (\exists m \geq n)\ m \in \mathrm{dom}(s) \wedge s(m) = f(m)\}$ is dense (by not finitely covering) and in N. □

Proof of Theorem II.15. From this lemma we get that forcing with $\mathbb{P}_{\mathcal{A}}$ produces an a.d. family $\mathcal{A} \cup \{g\}$ containing a function g that agrees on infinitely many inputs with each function in the ground model that is not finitely covered by $\mathcal{A}$.

$\mathbb{P}$ is the κ step finite support iteration of the $\mathbb{P}_{\mathcal{A}}$, where $\mathcal{A}$ at step α consists of the generics added so far ($\mathcal{A}_\alpha = \{g_\beta : \beta < \alpha\}$).

For any α, $|\mathcal{A}_\alpha| = |\alpha|$. So we can use as underlying set for $\mathbb{P}_{\mathcal{A}_\alpha}$ the set of $\langle s, H\rangle$, $s : \mathbb{N} \rightharpoonup \mathbb{N}$ finite partial and $H \subseteq |\alpha|$ $(= |\mathcal{A}|)$ finite. This shows that $|\mathbb{P}_{\mathcal{A}_\alpha}| = \max\{\omega, |\alpha|\}$, from which $|\mathbb{P}| = \kappa$ follows: since we use a finite support iteration of length κ we can use finite subsets of κ to indicate at which indices the element is non-maximal, and then for each of those an element from $\bigcup_{\alpha<\kappa} \mathbb{P}_{\mathcal{A}_\alpha}$ which is in the right coordinate. That is we can take the underlying set of $\mathbb{P}$ to be $\{p \in (\bigcup_{\alpha<\kappa} \mathbb{P}_{\mathcal{A}_\alpha})^{[\kappa]^{<\omega}} : \forall\alpha \in \mathrm{dom}(p)\ p(\alpha) \in \mathbb{P}_{\mathcal{A}_\alpha}\}$ (this also shows with the previous comment that the poset can be taken to have its underlying set in the ground model).

Since $\mathbb{P}$ is a finite support iteration of c.c.c. posets it preserves cardinals, and using the c.c.c. and that $|\mathbb{P}| = \kappa$ and $\kappa^\omega = 2^{\aleph_0}$ it follows that in the forcing extension $2^{\aleph_0}$ is still λ.

Now we see that the resulting collection $\mathcal{A} = \{g_\alpha : \alpha < \kappa\}$ is very mad as follows.

Let F, $|F| < |\mathcal{A}| = \kappa$, be a collection of functions not finitely covered by $\mathcal{A}$ (so also not finitely covered by any subcollection of $\mathcal{A}$). Since the forcing is finite support and $|F| < \mathrm{cf}(\kappa) = \kappa$, F already appears at some stage before κ. Then the generic added at any later stage will agree on infinitely many inputs with each member of F (by the lemma). □

2.5 Very Mad Families Can be Coanalytic

In this section we prove the following theorem.

Theorem II.17. *The Axiom of Constructibility implies the existence of a $\mathbf{\Pi}^1_1$ very mad family.*

The proof is based on the proof of the analogous result for maximal almost disjoint families of subsets of $\mathbb{N}$ by Arnold Miller, see [28]. For background on constructibility see [22, Chap. VI], and [11] in combination with [25] (the theory Basic Set Theory is not strong enough for the use Devlin makes of it, this is analyzed in Mathias paper, and a replacement is offered there that is sufficient for the results we use).

The idea of this proof is that we identify a set of good levels of L (those for which $L_\alpha \cong \mathrm{Sk}(L_\alpha)$ with small witness, as defined below). We prove a coding lemma

(Lemma II.18) allowing us to encode these levels into our construction. Then we show that from an encoding of a good level we have access to the limit level after it (Lemma II.23), which allows us to decide membership (Lemma II.24).

In this section we choose the sequence coding $\langle \ldots \rangle$ and projections π_i to be recursive.

Lemma II.18. *Let $A = \{g_n : n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ be an almost disjoint family, $E \subseteq \mathbb{N} \times \mathbb{N}$ and $F = \{f_n : n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ not finitely covered by A. Then there exists a function $g : \mathbb{N} \to \mathbb{N}$ almost disjoint from all functions in A, such that E is recursive in g and g agrees on infinitely many inputs with each member of F ($\forall n \in \mathbb{N}\, |f_n \cap g| = \aleph_0$).*

Proof. Instead of encoding E directly we encode χ the characteristic function of $\{\langle n, m\rangle : (n, m) \in E\}$.

We define g recursively. At step s we extend the initial segment of $\mathbb{N}$ on which g is defined by doing the following:

1. Find $n_{s,i}$, $i \in [0, s]$, such that $n_s < n_{s,0} < n_{s,1} < \cdots < n_{s,s}$, where n_s is the least number where g is not defined yet, and $f_i(n_{s,i})$ is different from all $g_0(n_{s,i}), \ldots, g_s(n_{s,i})$. Then define $g(n_{s,i}) = f_i(n_{s,i})$. Also define n_{s+1} to be $n_{s,s} + 1$.

2. Define $g(l)$ for $n_s < l < n_{s+1}$ where g is not yet defined to be the least number different from all $g_0(l), \ldots, g_s(l)$.

3. Define $g(n_s)$ to be $\langle k, \langle n_{s+1}, \chi(s)\rangle\rangle$ where k is the least number such that $\langle k, \langle n_{s+1}, \chi(s)\rangle\rangle$ is different from all $g_0(n_s), \ldots, g_s(n_s)$. Here the value n_{s+1} is the "pointer" to the next location where a value of χ can be found.

It can now be easily checked that the g constructed satisfies the lemma. □

We note that if A, F, E are members of L_α, then g is a member of $L_{\alpha+1}$; the proof shows how to define g from A, F and χ; also E and χ appear at the same level of the constructible hierarchy. Also note that the encoding is uniform: it does not depend on which functions and families we work with. This also means that we can talk about *the* relation encoded in g (later this relation will be the inclusion relation of a model on $(\mathbb{N}, E)$).

Definition II.19. For $\alpha > \omega$ we say $L_\alpha \cong \mathrm{Sk}(L_\alpha)$ iff there exists $\langle h, \varphi, \bar{p}\rangle$ (the witness) such that:

1. h is a Skolem function for all Σ_k formulas for L_α, for some $k \geq 1$,
2. $h[\mathbb{N} \times (\mathbb{N} \cup \bar{p})] \cong L_\alpha$, and
3. $h(n, x) = y \Leftrightarrow L_\alpha \models \varphi(\bar{p}, n, x, y)$.

Lemma II.20. *The set* $\{\alpha : L_\alpha \cong \mathrm{Sk}(L_\alpha)\}$ *is unbounded in* ω_1.

Proof. First recall from Gödel's proof of CH in L that every constructible real is in L_{ω_1}. From this using the fact all L_β, $\beta < \omega_1$, are countable it follows that the set $\{\beta < \omega_1 : \exists r\ [\, r \in L_{\beta+1} \setminus L_\beta \wedge r \in {}^{\mathbb{N}}\mathbb{N}\,]\}$ is unbounded in ω_1. So it is sufficient to prove for each β in this set that $L_{\beta+\omega} \cong \mathrm{Sk}(L_{\beta+\omega})$.

Therefore let r be definable over L_β from a finite sequence of parameters $\bar{q}$, $r = \{\langle m, n\rangle : L_\beta \models \psi(m, n, \bar{q})\}$, and such that $r \notin L_\beta$. Then $r \in L_{\beta+\omega}$ so that $L_{\beta+\omega} \models \exists r\ \forall m, n \in \omega\, \big((m, n) \in r \leftrightarrow \psi^{L_\beta}(m, n, \bar{q})\big)$.

Let $h : \mathbb{N} \times L_{\beta+\omega} \to L_{\beta+\omega}$ be a definable Skolem function for Σ_{k+2} formulas with $k \in \mathbb{N}$ such that $\psi \in \Sigma_k$.

As $X = h[\mathbb{N} \times (\mathbb{N} \cup \bar{q} \cup \{L_\beta\})] \prec_{k+2} L_{\beta+\omega}$ we have $(X, \in) \models \psi^{L_\beta}(n, m, \bar{q})$ iff $(L_\beta, \in) \models \psi^{L_\beta}(n, m, \bar{q})$ and $(X, \in) \models \exists r\ \forall m, n \in \omega\, \big((m, n) \in r \leftrightarrow \psi^{L_\beta}(m, n, \bar{q}\big)$, which shows r is in $(X, \in)$.

By the condensation lemma [11, Theorem II.5.2] we have a π such that $\pi : (X, \in) \cong (L_\alpha, \in)$, $\alpha \leq \beta+\omega$ and α is a limit ordinal; this π is the identity on transitive sets, in particular on the natural numbers. From this we get $(X, \in) \models \psi^{L_\beta}(n, m, \bar{q})$ iff $(L_\alpha, \in) \models \psi^{\pi L_\beta}(n, m, \pi\bar{q})$ and $(L_\alpha, \in) \models \exists r\, \forall m, n \in \omega\, \big((m, n) \in r \leftrightarrow \psi^{\pi L_\beta}(m, n, \pi\bar{q}\big)$, which shows that r is in L_α. So since $\alpha \leq \beta + \omega$, $r \notin L_\beta$, $r \in L_\alpha$, and α is a limit ordinal, we get $\alpha = \beta + \omega$. This means $X \cong L_{\beta+\omega}$ as was to be shown. □

Definition II.21. $\langle h, \varphi, \bar{p}\rangle$ is a *small witness* for $L_\alpha \cong \mathrm{Sk}(L_\alpha)$ if it is a witness for $L_\alpha \cong \mathrm{Sk}(L_\alpha)$ and there exist r, ψ, and β such that

1. L_β is in $\bar{p}$;
2. If $\bar{q}$ is $\bar{p}$ with L_β removed, then $r = \{(m, n) : L_\beta \models \psi(m, n, \bar{q})\}$;
3. $\alpha = \beta + \omega$;
4. h is a Skolem function for Σ_{k+2} formulas for L_α and k such that $\psi \in \Sigma_k$.

That is $\langle h, \varphi, \bar{p}\rangle$ is related to r and β as in the proof of the previous lemma.

The reason for this definition is that if we have a small witness, we can check in $L_{\alpha+\omega}$ that $\langle h, \varphi, \bar{p}\rangle$ is obtained from r as in the proof of the lemma. For a general witness we don't know at which level of L the isomorphism $h[\mathbb{N} \times (\mathbb{N} \cup p)] \cong L_\alpha$ will appear.

The proof of the lemma shows that the set $\{\alpha < \omega_1 : L_\alpha \cong \mathrm{Sk}(L_\alpha)$ with a small witness$\}$ is unbounded. Enumerate it in increasing order by $\langle \beta_\gamma : \gamma < \omega_1\rangle$. Note that by absoluteness of the notion of small witness and the fact that limit levels of the constructible hierarchy are closed under certain simple recursions, we have that $L_{\beta_\gamma+\omega} \models$"$\langle \beta_{\gamma'} : \gamma' \leq \gamma\rangle$ is an initial segment of the increasing enumeration of ordinals α such that $L_\alpha \cong \mathrm{Sk}(L_\alpha)$ with small witness".

Lemma II.22. *If $L_\alpha \cong \mathrm{Sk}(L_\alpha)$, then there is an $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $E \in L_{\alpha+\omega}$ and $(L_\alpha, \in) \cong (\mathbb{N}, E)$.*

Proof. Let $L_\alpha \cong \mathrm{Sk}(L_\alpha)$ be witnessed by $\langle h, \varphi, \bar{p}\rangle$. First notice that $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle) \in L_{\alpha+\omega}$: we follow the ideas from pages 40 and 41 of [11]. The theorem Devlin proves there is not correct, see [25], but the method can be used here. We have a function f such that $f(0)$ is the set of all primitive formulas of set theory, and $f(i+1)$ is the set of all formulas formed from the formulas in $f(i)$ by conjunction, disjunction, implication, and quantification. Then we construct a function g such that $g(i)$ is a set of pairs, first coordinate a formula φ from $f(i)$, second coordinate a sequence $\bar{x}$ of elements of L_α such that $\varphi(\bar{x})$ is true in $(L_\alpha, \in, \bar{p})$. All these elements are in $L_{\alpha+n}$ for some n. Then at $L_{\alpha+n+1}$ we can construct all $g \restriction k$ for $k \in \mathbb{N}$. So at $L_{\alpha+n+2}$ we can use the recursive definition of g to construct it. From g we get $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ as the subset of the image consisting of all formulas with no free variables. (Note that this, and the following, are all uniform with respect to the sequence $\bar{p}$, but, for notational convenience, we'll leave it implicit as a parameter.)

Let $e : \mathbb{N} \to \mathbb{N} \times (\mathbb{N} \cup \bar{p})$ be the definable bijection

$$e(n) = \begin{cases} (\pi_0(n), \bar{p}_{\pi_1(n)}), & \text{if } \pi_1(n) < \mathrm{lh}(\bar{p}); \\ (\pi_0(n), \pi_1(n) - \mathrm{lh}(\bar{p})), & \text{otherwise,} \end{cases}$$

and φ_e the formula defining e, i.e. $\varphi_e(n, x, y) \Leftrightarrow e(n) = (x, y)$ (this formula defines e in any $L_{\alpha+4}$ with $\alpha > \omega$ and $\bar{p} \in L_\alpha$ and is absolute for these levels).

Define $\tilde{e} : \mathbb{N} \to \mathbb{N}$ from this by setting $\tilde{e}(0) = 0$ and $\tilde{e}(n+1) = k$ where k is the least number bigger than $\tilde{e}(n)$ such that $\ulcorner \psi(k, \tilde{e}(n)) \urcorner \in \mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$, where

$\psi(k, \tilde{e}(n))$ is the formula

$$\forall l \leq \tilde{e}(n)\ \forall y_0, y_1\ \big[\ \varphi(\bar{p}, \pi_0(e(l)), \pi_1(e(l)), y_0)\ \wedge\ \varphi(\bar{p}, \pi_0(e(k)), \pi_1(e(k)), y_1) \rightarrow$$
$$\big(\exists z\ (z \in y_0 \wedge z \notin y_1)\ \vee\ (z \notin y_0 \wedge z \in y_1)\big)\ \big],$$

in which φ is the formula defining h, and which after elimination of e in favor of its definition becomes

$$\forall l \leq \tilde{e}(n)\ \forall l_0, l_1, k_0, k_1\ \big\{\ \varphi_e(l, l_0, l_1)\ \wedge\ \varphi_e(k, k_0, k_1) \rightarrow$$
$$\forall y_0, y_1\ \big[\ \varphi(\bar{p}, l_0, l_1, y_0)\ \wedge\ \varphi(\bar{p}, k_0, k_1, y_1) \rightarrow$$
$$\big(\ \exists z\ (z \in y_0 \wedge z \notin y_1)\ \vee\ (z \notin y_0 \wedge z \in y_1)\big)\ \big]\ \big\}.$$

Note $\psi(k, \tilde{e}(n))$ is the formula expressing $\forall l \leq \tilde{e}(n)\ h(e(k)) \neq h(e(l))$, and a Gödel number for $\psi(k, \tilde{e}(n))$ can be recursively obtained from k and n (the function $(k, m) \mapsto \ulcorner \forall x \theta_m(x) \rightarrow \psi(k, x) \urcorner$ (where $\theta_m(x)$ is the formula defining $m \in \omega$) is in $L_{\omega+\omega}$, but $\tilde{e}$ which is recursively defined from it and $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ can be constructed at the level of L after $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ is constructed).

Let $\varphi_{\tilde{e}}(n, m)$ be such that $\varphi_{\tilde{e}}(n, m) \Leftrightarrow \tilde{e}(n) = m$.

These definitions have been made so that $h \circ e \circ \tilde{e} : \mathbb{N} \rightarrow L_\alpha$ is an enumeration of $h[\mathbb{N} \times (\mathbb{N} \cup \bar{p})]$ without repetitions. We will set up the model $(\mathbb{N}, E)$ such that the number $m \in \mathbb{N}$ will represent the set $h(e(\tilde{e}(m)))$. It is then clear that we want $n\ E\ m$ iff $h(e(\tilde{e}(n))) \in h(e(\tilde{e}(m)))$.

We show $E \in L_{\alpha+k}$ for some $k \in \mathbb{N}$ by eliminating all functions in favor of their definitions in the statement $h(e(\tilde{e}(n))) \in h(e(\tilde{e}(m)))$, and then noting this statement is true of (n, m) iff the Gödel number of the formula resulting from substituting terms defining n and m in this formula is in $\mathrm{Th}(\langle L_\alpha, E, \bar{p}\rangle)$.

First eliminating h, we get

$$\forall z_n, z_m \left[\left\{ \varphi(\bar{p}, \pi_0(e(\tilde{e}(n))), \pi_1(e(\tilde{e}(n))), z_n) \wedge \right.\right.$$
$$\left.\left. \varphi(\bar{p}, \pi_0(e(\tilde{e}(m))), \pi_1(e(\tilde{e}(m))), z_m) \right\} \rightarrow z_n \in z_m \right].$$

Then eliminating e we get

$$\forall x_n, y_n, x_m, y_m \left\{ \varphi_e(\tilde{e}(n), x_n, y_n) \ \wedge \ \varphi_e(\tilde{e}(m), x_m, y_m) \rightarrow \right.$$
$$\left. \forall z_n, z_m \left[\ \varphi(\bar{p}, x_n, y_n, z_n) \ \wedge \ \varphi(\bar{p}, x_m, y_m, z_m) \rightarrow z_n \in z_m \right] \right\}.$$

After eliminating $\tilde{e}$ this gives

$$\forall l_n, l_m \left(\ \varphi_{\tilde{e}}(n, l_n) \ \wedge \ \varphi_{\tilde{e}}(m, l_m) \rightarrow \right.$$
$$\forall x_n, y_n, x_m, y_m \left\{ \ \varphi_e(l_n, x_n, y_n) \ \wedge \ \varphi_e(l_m, x_m, y_m) \rightarrow \right.$$
$$\left.\left. \forall z_n, z_m \left[\ \varphi(\bar{p}, x_n, y_n, z_n) \ \wedge \ \varphi(\bar{p}, x_m, y_m, z_m) \rightarrow z_n \in z_m \right] \right\} \right).$$

This is a formula in the language $\{\in, \bar{p}\}$ with free variables n and m. The recursive function G that to (n, m) assigns the formula

$$\forall u, v \ \theta_n(u) \wedge \theta_m(v) \rightarrow$$
$$\forall l_n, l_m \left(\ \varphi_{\tilde{e}}(u, l_n) \ \wedge \ \varphi_{\tilde{e}}(v, l_m) \rightarrow \right.$$
$$\forall x_n, y_n, x_m, y_m \left\{ \ \varphi_e(l_n, x_n, y_n) \ \wedge \ \varphi_e(l_m, x_m, y_m) \rightarrow \right.$$
$$\left.\left. \forall z_n, z_m \left[\ \varphi(\bar{p}, x_n, y_n, z_n) \ \wedge \ \varphi(\bar{p}, x_m, y_m, z_m) \rightarrow z_n \in z_m \right] \right\} \right).$$

is in $L_{\alpha+l}$ for some $l \in \mathbb{N}$ (note: $\varphi_{\tilde{e}}$ uses $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ as a parameter).

This shows we can define E over $L_{\alpha+l}$ by $(n, m) \in E$ iff $G(n, m) \in \mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$.

□

We now define functions (as in [14, page 217]) relating the natural numbers and the real numbers to their representatives in $(\mathbb{N}, E)$.

Define for any $(\mathbb{N}, E) \cong L_\alpha$, $\omega < \alpha < \omega_1$, a recursive function $\mathsf{Nat}_E : \mathbb{N} \to \mathbb{N}$ by

$$\mathsf{Nat}_E(0) = \text{the unique } u \in \mathbb{N} \text{ such that } \forall l \in \mathbb{N}\ (\neg\, l\ E\ u)$$

$$\mathsf{Nat}_E(n+1) = \text{the unique } u \in \mathbb{N} \text{ such that } \forall l \in \mathbb{N}[(l\ E\ u) \leftrightarrow \big((l\ E\ \mathsf{Nat}_E(n)) \vee (l = \mathsf{Nat}_E(n))\big)]$$

Using this we can define $\mathsf{Real}_E : {}^{\mathbb{N}}\mathbb{N} \to \mathbb{N}$ a partial function by

$$\mathsf{Real}_E(r) = \text{the unique (if exists) } u \in \mathbb{N} \text{ such that } \forall n, m[r(n) = m \leftrightarrow (\mathbb{N}, E) \models u(\mathsf{Nat}_E(n)) = \mathsf{Nat}_E(m)]$$

If $L_\alpha \cong \mathrm{Sk}(L_\alpha)$, then there exists $\pi : (L_\alpha, \in) \cong (\mathbb{N}, E)$. So the sets $\mathbb{R}_\alpha = {}^{\mathbb{N}}\mathbb{N} \cap L_\alpha$ and $\mathbb{R}_E := \{n \in \mathbb{N} : (\mathbb{N}, E) \models n \text{ is a real}\}$ are mapped to each other by the isomorphism. We have in fact that if $r \in \mathbb{R}_\alpha$ then $r(k) = l$ iff $\pi(r)(\mathsf{Nat}_E(k)) = \mathsf{Nat}_E(l)$ is true in $(\mathbb{N}, E)$. So we can define in $L_{\alpha+\omega}$ an enumeration $e_\alpha : \mathbb{N} \to R_\alpha$ of all reals in L_α as follows:

First let $e : \mathbb{N} \to \mathbb{R}_E$ be the bijection $e(0) = \min\{\mathbb{R}_E\}$ and $e(n+1) = \min\{m \in \mathbb{R}_E : m > e(n)\}$. Then e_α is e composed with the map defined by

$$\{(n, r) \in \mathbb{R}_E \times \mathbb{R}_\alpha : \forall k, l \in \mathbb{N}\ (\mathbb{N}, E) \models n(\mathsf{Nat}_E(k)) = \mathsf{Nat}_E(l) \leftrightarrow r(k) = l\} = \{(n, r) \in \mathbb{R}_E \times \mathbb{R}_\alpha : \pi(r) = n\}.$$

Now we are ready for the construction of the very mad family $\mathcal{A}$ which we will show is coanalytic. It will be recursively enumerated as $\langle g_\alpha : \alpha < \omega_1 \rangle$.

To define g_γ from $\langle g_\alpha : \alpha < \gamma \rangle$ we use Lemma II.18 with $A = A_\gamma = \langle g'_n : n \in \mathbb{N} \rangle$, $F = F_\gamma = \langle f_n : n \in \mathbb{N} \rangle$ and E as described below.

By induction we will have the set $\{g_\alpha : \alpha < \gamma\}$ in $L_{\beta_\gamma+\omega}$ (β_γ as defined on page 34), and by a recursion in $L_{\beta_\gamma+\omega}$ we get the enumeration $\langle g_{\gamma'} : \gamma' < \gamma \rangle$ in $L_{\beta_\gamma+\omega}$. We can recursively find an enumeration $\langle g'_n : n \in \mathbb{N}\rangle$ of it in $L_{\beta_\gamma+\omega}$ by letting g'_n be the n^{th} member in the enumeration e_{β_γ} of $\mathbb{R}_{\beta_\gamma}$ which is in $\{g_\alpha : \alpha < \gamma\}$.

We then recursively define f_n to be the n^{th} member in the enumeration of $\mathbb{R}_{\beta_\gamma}$ which is not finitely covered by $\{g_\alpha : \alpha < \gamma\}$. This enumeration will also be in $L_{\beta_\gamma+\omega}$.

By Lemma II.22 we have an E such that $(\mathbb{N}, E) \cong (L_{\beta_\gamma}, \in)$ in $L_{\beta_\gamma+\omega}$.

After application of Lemma II.18 (and the observation following it) we get $g_\gamma \in L_{\beta_\gamma+\omega}$. This finishes the construction. Note that this construction is absolute for $L_{\beta_\gamma+\omega}$.

Clearly $\mathcal{A}$ is an a.d. family, and if $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ with $|F| < |A| = \aleph_1$, then there is a $\beta < \omega_1$ such that $F \subseteq L_\beta$. Now if F is not finitely covered by $\mathcal{A}$ then for every $f \in F$ and every γ with $\beta_\gamma \geq \beta$ the set $f \cap g_\gamma$ is infinite, which shows that $\mathcal{A}$ is a very mad family.

Now what remains to be seen is that this $\mathcal{A}$ is Π^1_1 definable.

Lemma II.23. *If $(\mathbb{N}, E) \cong (L_\alpha, \in)$ and $g \in L_{\alpha+\omega}$ encodes E as in Lemma II.18, then there is a formula φ only containing quantifiers over the natural numbers such that*

$$\begin{aligned}\varphi(\langle E_\omega, r, u\rangle, g) \Leftrightarrow (\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in)\ \wedge \\ r \textit{ is the satisfaotion relation for } (\mathbb{N}, E_\omega)\ \wedge \\ u = \mathsf{E}\end{aligned}$$

where:

Sat: The formula $\mathsf{Sat}(E_\omega, r)$ states that r is the satisfaction relation for E_ω: (sketch)

$$\begin{aligned} r(\langle\ulcorner\varphi\urcorner,\bar{m}\rangle)=1 \Leftrightarrow & (\ulcorner\varphi\urcorner=\ulcorner x=y\urcorner \wedge m_0=m_1) \vee \\ & (\ulcorner\varphi\urcorner=\ulcorner x\in y\urcorner \wedge E_\omega(m_0,m_1)) \vee \\ & (\ulcorner\varphi\urcorner=\ulcorner\forall x\psi(x)\urcorner \wedge \forall n\, r(\langle\ulcorner\psi\urcorner,\langle n,\bar{m}\rangle\rangle)=1) \vee \\ & (\ulcorner\varphi\urcorner=\ulcorner\neg\psi\urcorner \wedge r(\langle\ulcorner\psi\urcorner,\bar{m}\rangle)=0) \vee \\ & (\ulcorner\varphi\urcorner=\ulcorner\psi_1\vee\psi_2\urcorner \wedge (r(\langle\ulcorner\psi_1\urcorner,\bar{m}\rangle)=1 \vee r(\langle\ulcorner\psi_2\urcorner,\bar{m}\rangle)=1)) \end{aligned}$$

$\mathsf{EonEvens}$: $\mathsf{EonEvens}(E_\omega, E)$ states that E is isomorphic to an initial segment of E_ω and lives on the even natural numbers.

$$\mathsf{EonEvens}(E_\omega, E) \equiv \forall i,j\,\big(\neg(2i+1\ E_\omega\ 2j)\ \wedge\ (2i\ E_\omega\ 2j \leftrightarrow i\ E\ j)\big)$$

Levels: Here we need a bijection $\pi:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that $\pi(0,0)=1$ and $\pi(0,k+1)$ enumerates the evens; we can easily find such a bijection which is recursive.

Then $\mathsf{Levels}(E_\omega, E, r)$ is the conjunction of $\mathsf{SLevels}(E_\omega, E)$ and $\mathsf{ELevels}(E_\omega, E, r)$ where $\mathsf{SLevels}$ states $\pi(l,0)$ is the l-th level after $(\mathbb{N}, E)$:

$$\begin{aligned} &\forall l,i,j\,\big(\,\big[\,i<l \to \pi(i,j)\ E_\omega\ \pi(l,0)\,\big]\ \wedge\ \pi(l,j+1)\ E_\omega\ \pi(l,0)\,\big)\wedge \\ &\forall l,i,j\,\big(\pi(i,j)\ E_\omega\ \pi(l,0) \to (i<l \vee (i=l \wedge j>1))\big), \end{aligned}$$

and $\mathsf{ELevels}(E_\omega, E, r)$ that $k\mapsto\pi(l,k+1)$ is an enumeration of the new sets at the level l^{th} after $(\mathbb{N}, E)$. First we find an enumeration, $k\mapsto\mathsf{ge}(l,k)$, of formulas and parameters that can be used to define sets at the l^{th} level:

Let S be the set $\{(n,\bar{x}) : n$ is the Gödel number of a formula with $\mathrm{lh}(\bar{x})+1$ free variables $\wedge\ \bar{x}\in {}^{<\mathbb{N}}\mathbb{N}\}$. Then define $\mathsf{ge}:\mathbb{N}\times\mathbb{N}\to S$ such that $\mathsf{ge}[\{(l,k):k\in\mathbb{N}\}] =$

$\{(n, \bar{x}) \in S : \bar{x} \in {}^{<\mathbb{N}}(\{\pi(l, k+1) : k \in \mathbb{N}\} \cup \{\pi(j, k) : j < l \wedge k \in \mathbb{N}\})\}$. Such a function ge can clearly be chosen to be recursive.

We want to define $\tilde{\mathsf{ge}} : \mathbb{N} \times \mathbb{N} \to S$ such that $k \mapsto \tilde{\mathsf{ge}}(l, k)$ enumerates only the data needed to define new sets at level l, and does so without repetition. For this we do some preliminary work.

First note that $(\pi(l, 0), E_\omega) \models \varphi(x)$ is equivalent to $(\mathbb{N}, E_\omega) \models (\varphi(x))^{\pi(l,0)}$ which in turn is equivalent to $r(\langle \ulcorner(\varphi)^{\pi(l,0)}\urcorner, x\rangle = 1$. The map $\mathsf{rel} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $(\ulcorner\varphi\urcorner, l) \mapsto \ulcorner(\varphi)^{\pi(l,0)}\urcorner$ and 0 if the first component of the input is not the Gödel number of a formula is recursive.

Then we define a formula $\mathsf{new}(n, \bar{x}, l)$ such that it is true of $(n, \bar{x}, l)$ iff $n = \ulcorner\varphi\urcorner$ and $\{y \; E_\omega \; \pi(l, 0) : (\pi(l, 0), E_\omega) \models \varphi(\bar{x}, y)\}$ is different from all $\pi(j, k)$ for $j \le l$ and $k \in \mathbb{N}$. This means that the set determined by $(n, \bar{x})$ at level l didn't exist before level l (and is not the collection of all sets before level l, which is $\pi(l, 0)$). The formula expressing this is:

$$\begin{aligned}\mathsf{new}(n, \bar{x}, l) \equiv \forall j, k \; j \le l \to \exists j', k' \big\{ j' \le l \wedge \\ \big[\big(\pi(j', k') \; E_\omega \; \pi(j, k) \wedge r(\langle \mathsf{rel}(n, l), \langle x, \pi(j', k')\rangle\rangle) = 0 \big) \vee \\ \big(\neg\pi(j', k') \; E_\omega \; \pi(j, k) \wedge r(\langle \mathsf{rel}(n, l), \langle x, \pi(j', k')\rangle\rangle) = 1 \big) \big] \big\}\end{aligned}$$

We also need a formula $\mathsf{nb}(l, m)$ that is true of (l, m) iff the set defined by $\mathsf{ge}(l, m)$

from $\pi(l,0)$ is not also defined by $\mathsf{ge}(l,m')$ with $m'<m$.

$$\begin{aligned}\mathsf{nb}(l,m) \equiv \forall m' \leq m \exists j,k(j<l \vee (j=l \wedge k>0)) \wedge \\ \Big[\big(r(\langle \mathsf{rel}(n,l), \langle x, \pi(j,k)\rangle\rangle) = 1 \wedge \\ r(\langle \mathsf{rel}(\pi_0(\mathsf{ge}(l,m')),l), \langle \pi_1(\mathsf{ge}(l,m')), \pi(j,k)\rangle\rangle) = 0 \big) \vee \\ \big(r(\langle \mathsf{rel}(n,l), \langle x, \pi(j,k)\rangle\rangle) = 0 \wedge \\ r(\langle \mathsf{rel}(\pi_0(\mathsf{ge}(l,m')),l), \langle \pi_1(\mathsf{ge}(l,m')), \pi(j,k)\rangle\rangle) = 1 \big) \Big]\end{aligned}$$

Now we can define $\tilde{\mathsf{ge}}$:

$$\begin{aligned}\tilde{\mathsf{ge}}(l,0) &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such that} \\ &\quad (n,x) = \mathsf{ge}(l,k) \text{ defines a new set} \\ &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such that for} \\ &\quad (n,x) = \mathsf{ge}(l,k) \text{ we have } \mathsf{new}(n,x,l)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathsf{ge}}(l,m+1) &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such that } (n,x) = \mathsf{ge}(l,k) \\ &\quad \text{defines a new set that is not already defined by } \mathsf{ge}(l,\tilde{k}) \\ &\quad \text{with } \tilde{k} \text{ less than the } k \text{ used in } \tilde{\mathsf{ge}}(l,m) \\ &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such that for } (n,x) = \mathsf{ge}(l,k) \\ &\quad \text{we have } \mathsf{new}(n,x,l) \wedge \mathsf{nb}(l,k)\end{aligned}$$

Now the formula $\mathsf{ELevels}$ can be defined:

$$\forall l,k[\pi(l+1,k+1) \text{ is defined from } \pi(l,0)$$
$$\text{by the formula and parameters in } \tilde{\mathsf{ge}}(l,k)]$$
$$\Leftrightarrow \forall l,k,n,x((n,x)=\tilde{\mathsf{ge}}(l,k)\to$$
$$[\forall y r(\langle \mathsf{rel}(n,l),\langle x,y\rangle\rangle)=1 \leftrightarrow y\ E_\omega\ \pi(l,k)]$$

□

Note that with these formulas, if $(\mathbb{N},E)$ is wellfounded, then so is $(\mathbb{N},E_\omega)$ (which is the main reason for the lemma to be done the way it is).

Let $\xi_{\mathrm{s}}\in\Sigma_1$ and $\xi_{\mathrm{p}}\in\Pi_1$ be the formulas witnessing that the class $H=\{(x,\gamma): x=L_\gamma\}$ is uniformly $\Delta_1^{L_\alpha}$ for $\alpha>\omega$ a limit ordinal (see [11, Lemma 2.7]: the proof of this lemma uses some results from earlier in the book which are not correct, but in [25] (key result on p. 45) it is shown that there is a theory which is strong enough to prove these results and which is true at L_α for α a limit ordinal).

Let $E\subseteq\mathbb{N}\times\mathbb{N}$ be such that $(\mathbb{N},E)$ is wellfounded, and let r be its satisfaction relation. Then let $\chi(E,r)$ be the formula (we write $(\mathbb{N},E)\models\theta$ for $r(\ulcorner\theta\urcorner)$)

$$\forall n,m\in\mathbb{N}\ \big[\,(\mathbb{N},E)\models \text{“}n \text{ is an ordinal”}\ \to$$
$$(\mathbb{N},E)\models\xi_{\mathrm{p}}(m,n)\leftrightarrow(\mathbb{N},E)\models\xi_{\mathrm{s}}(m,n)\,\big]\wedge$$
$$(\mathbb{N},E)\models\ \text{“there is no largest ordinal”}\ \wedge$$
$$\exists n\in\mathbb{N}\ (\mathbb{N},E)\models\text{“}n=\omega\text{”}\ \wedge$$
$$(\mathbb{N},E)\models\forall x\exists y\,\big(\,y\text{ is an ordinal }\ \wedge\forall z\,(\xi_{\mathrm{p}}(z,y)\to x\in z)\,\big).$$

Then the image X of the Mostowski collapse of $(\mathbb{N},E)$ satisfies that H is Δ_1^X, there is no largest ordinal, $\omega\in X$, and $V=L$. This gives us that $X=(V)^X=(L)^X=L_\alpha$ for $\alpha=X\cap\mathsf{Ord}$ a limit ordinal $>\omega$.

Lemma II.24.

$$g \in \mathcal{A} \Leftrightarrow \textit{the model encoded in } g \textit{ is wellfounded} \wedge$$

$$\forall \langle E_\omega, r, u\rangle \ \varphi(\langle E_\omega, r, u\rangle, g) \wedge \chi(E_\omega, r) \rightarrow r(\ulcorner u \in \mathcal{A}\urcorner, \bar{\emptyset}) = 1.$$

Proof. By induction on $\gamma < \omega_1$ we show that for all reals in L_{β_γ} the equivalence holds. So assume that $g \in L_{\beta_\gamma}$ and for all $\gamma' < \gamma$ we have the equivalence for all reals in $L_{\beta_{\gamma'}}$.

If $g \in \mathcal{A}$, then g uniformly encodes $(\mathbb{N}, E)$ such that $(\mathbb{N}, E) \cong (L_{\beta_{\gamma'}}, \in)$ with $\gamma' < \gamma$. The unique model $(\mathbb{N}, E_\omega)$ satisfying $\varphi(\langle E_\omega, r, u\rangle, g)$ has $(\mathbb{N}, E_\omega) \cong (L_{\beta_{\gamma'}+\omega}, \in)$, so also satisfies χ. And in the description of the construction we have shown that $(L_{\beta'_\gamma+\omega}, \in) \models g \in \mathcal{A}$, i.e. $(\mathbb{N}, E_\omega) \models \ulcorner u \in \mathcal{A}\urcorner$ where u represents g in the model.

If the model encoded by g is wellfounded and we have $\forall \langle E_\omega, r, u\rangle \ \varphi(\langle E_\omega, r, u\rangle, g) \wedge \chi(E_\omega, r) \rightarrow r(\ulcorner u \in \mathcal{A}\urcorner, \bar{\emptyset}) = 1$, then the unique $\langle E_\omega, r, u\rangle$ for which $\varphi(\langle E_\omega, r, u\rangle, g)$ has that $(\mathbb{N}, E_\omega)$ is wellfounded and satisfies $\chi(E_\omega, r)$. So there is a countable limit $\beta > \omega$ such that $(\mathbb{N}, E_\omega) \cong (L_\beta, \in)$. Since $(\mathbb{N}, E_\omega) \models u \in \mathcal{A}$, we have $(L_\beta, \in) \models g \in \mathcal{A}$, which by absoluteness gives $g \in \mathcal{A}$. □

Since the formula on the right hand side of the equivalence is clearly Π^1_1, this completes the proof of the theorem.

CHAPTER III

Cofinitary Groups

In this chapter we prove our results on (maximal) cofinitary groups. We will again repeat the definitions for convenience, and give a short list of notational conventions used. After this, in Section 3.1, we give some of the basic ideas used in almost all constructions.

Definition III.1. (i). $\mathrm{Sym}(\mathbb{N})$ is the group of bijections of the natural numbers with group operation composition.

(ii). A bijection $g \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* if it has finitely many fixed points, or is the identity.

(iii). A subgroup $G \leq \mathrm{Sym}(\mathbb{N})$ is *cofinitary* if all of its members are cofinitary.

(iv). A subgroup $G \leq \mathrm{Sym}(\mathbb{N})$ is *maximal cofinitary* if it is a cofinitary group and is not properly contained in another cofinitary group.

Now we will give our guide to notation in this chapter.

G and H will be (maximal) cofinitary groups. g with sub- and superscripts will be elements of this group, elements under consideration to be added to the group, or finite approximations of elements thereof.

3.1 Basics

In this section we will explain the construction of a maximal cofinitary group from the continuum hypothesis. That there exists a maximal cofinitary group follows immediately from the wellorder of ${}^{\mathbb{N}}\mathbb{N}$ that exists under CH, but here we use it to introduce the basic ideas for constructing a maximal cofinitary group. This construction is from earlier work by Yi Zhang, see [37], and gives in a convenient setting the basic ideas used frequently later on.

We will construct the group by constructing a sequence of generators $\langle g_\alpha : \alpha < \omega_1 \rangle$, such that $\langle \{g_\alpha : \alpha < \omega_1\} \rangle$ (the group generated by the set $\{g_\alpha : \alpha < \omega_1\}$) is a maximal cofinitary group. This sequence will be constructed recursively.

The important step is adding to a given countable cofinitary group G a new generator g such that $\langle G, g \rangle$ is still cofinitary and iteration of this construction ω_1 many times gives a maximal cofinitary group.

We first examine how to ensure we get a cofinitary group after adding a new generator.

Definition III.2. (i). For G and H two groups, we write $G * H$ for the free product of G and H.

(ii). Define for a group $G \leq \mathrm{Sym}(\mathbb{N})$ the group W_G to be $G * F(x)$ with $F(x)$ the free group on the generator x.

If $w(x) \in W_G$ then $w(x)$ has the reduced form

$$(**) \qquad w(x) = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_{l-1}} g_l,$$

with $g_i \in G$ ($i \leq l$) and $k_i \in \mathbb{Z} \setminus \{0\}$ ($i \leq l - 1$).

Definition III.3. For $w(x)$ as in (**), its length $\mathrm{lh}(w)$ is defined to be $l + 1 + \Sigma_{i=0}^{l-1} k_i$.

The set W_G is useful in this context because of the following lemma.

Lemma III.4. *If $h \in \langle G, g\rangle$, then there is a $w(x) \in W_G$ such that $w(g) = h$.*

So for the group $\langle G, g\rangle$ to be cofinitary it suffices to ensure that each $w(g)$ has finitely many fixed points or is the identity.

The bijection g will be constructed recursively from finite approximations, i.e. $g = \bigcup_{s\in\mathbb{N}} g_s$ with $g_s : \mathbb{N} \rightharpoonup \mathbb{N}$ finite and injective. For this our strategy will be to avoid as many fixed points as possible.

Definition III.5. $z \in W_G$ is a conjugate subword of $w \in W_G$ if there exists a u such that without cancellation $w = u^{-1}zu$.

If for a conjugate subword z of w the partial permutation $z(g_s)$ has a fixed point, then for any extension to a total permutation g this fixed point will give rise to a fixed point of w. So the best we can hope for extending finite approximations is encoded in the following definition.

Definition III.6. For $w \in W_G$ and finite one-to-one functions p, q such that $p \subseteq q$, we say that q is a *good extension* of p with respect to w if the following condition is satisfied:
for each $l \in \mathbb{N}$ such that

$$w(p)(l) \text{ is undefined and } w(q)(l) = l,$$

there are subwords u and z of w and $n \in \mathbb{N}$ such that

$$w = u^{-1}zu \text{ without cancellation,}$$
$$u^{-1}(q)(n) = l, \text{ and } z(p)(n) = n.$$

The following lemma shows that this definition works.

Lemma III.7. *If G is any countable cofinitary group, $w(x) \in W_G$, and $g = \bigcup_{s \in \mathbb{N}} g_s$ where all g_s are finite and for some t and all $s \geq t$, g_{s+1} is a good extension of g_s with respect to w and all of its subwords, then $w(g)$ will have finitely many fixed points.*

We prove the following more quantitative version.

Lemma III.8. *In the context of the lemma above $w(g)$ will have the same number of fixed points as $z(g_t)$, where z is the shortest conjugate subword of w.*

We use the following definitions related to a word $w \in W_G$; these are also used frequently in later sections.

Definition III.9. (i). Define $w_{(i)}$ to be the i^{th} letter in w counted from the right (for example if $w = g_0 x^2 g_1$, then $w_{(0)} = g_1$, $w_{(1)} = x$, $w_{(2)} = x$, and $w_{(4)} = w_{(\mathrm{lh}(w))} = g_0$).

(ii). For $p : \mathbb{N} \rightharpoonup \mathbb{N}$ a partial function, $w(x) \in W_G$ and $n \in \mathbb{N}$, we define the *evaluation path* for n in $w(p)$ to be the sequence $\langle l_i \in \mathbb{N} : i \leq j \rangle$, with $l_0 := n$, $l_{i+1} := w_i(p)(l_i)$ and

$$j := \begin{cases} \mathrm{lh}(w), & \text{if } w(p)(l) \text{ is defined;} \\ \max\{i : w_i(p)(l_i) \text{ is defined}\} + 1, & \text{otherwise.} \end{cases}$$

(iii). Define $w \restriction i$ to be $w_{(i-1)} w_{(i-2)} \cdots w_{(0)}$, the initial segment (from the right) of length i of the word w. The evaluation path for n can be expressed then as $l_i = (w \restriction i)(p)(n)$

(iv). The pairs (l_i, l_{i+1}) of p are the pairs of p *used* in this evaluation. For a general function f (possibly partial) we call $(n, f(n))$ a pair from f.

Proof of Lemma III.8. By case analysis:

Case 1: w has only one conjugate subword (itself). Then $w(g)$ will have as many subwords as $w(g_t)$. Since there are no proper conjugate subwords, in the definition of good extension every time z will equal w. It is then clear no new fixed points can be introduced.

Case 2: w has more than one conjugate subword. Let z be the minimal conjugate subword. Notice that in any expression $w = u^{-1}z'u$ we have that z is a conjugate subword of z', so we can divide further. A fixed point of z' at some stage s "comes from" a fixed point of z defined at that same stage: let $z'(g_s)(n) = n$, and $z' = v^{-1}zv$. Then $z = vz'v^{-1}$ and $(vz'v^{-1})(g_s)(v(g_s)(n)) = (vz(g_s))(n) = v(g_s)(n)$. And by the previous case, the total number of fixed points of $z(g)$ is equal to the total number of fixed points of $z(g_t)$. □

The nice and amazing thing is that enough good extensions exist to achieve what we need. This was shown with the following lemma(s) by Yi Zhang, see [34] and [37].

Lemma III.10. *If G is a countable cofinitary group, p a finite injective function $\mathbb{N} \rightharpoonup \mathbb{N}$, and $w \in W_G$ then*

*(i). **(Domain Extension Lemma)** for all $n \in \mathbb{N} \setminus \mathrm{dom}(p)$, for all but finitely many $k \in \mathbb{N}$, the extension $p \cup \{(n, k)\}$ is a good extension of p with respect to w.*

*(ii). **(Range Extension Lemma)** for all $k \in \mathbb{N} \setminus \mathrm{ran}(p)$, for all but finitely many $n \in \mathbb{N}$ the extension $p \cup \{(n, k)\}$ is a good extension of p with respect to w.*

*(iii). **(Hitting f Lemma)** for all $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f\rangle$ is cofinitary, for all but finitely many $n \in \mathbb{N}$ the extension $p \cup \{(n, f(n))\}$ is a good extension of p with respect to w.*

Since in each of the items in this lemma there are only finitely many choices for a particular word to get an extension that is not good, these items are also true for finite lists of words. Also, as is clear from the proof, the hitting f lemma also works for f that are infinite partial functions.

With these preparations we can prove that the continuum hypothesis implies the existence of a maximal cofinitary group. With the continuum hypothesis enumerate $\mathrm{Sym}(\mathbb{N})$ as $\langle f_\alpha : \alpha < \omega_1 \rangle$.

We construct a generating set of this group recursively as $\langle g_\alpha : \alpha < \omega_1 \rangle$. At step β we have already constructed $\langle g_\alpha : \alpha < \beta \rangle$, generating a countable cofinitary group $G_\beta = \langle \{g_\alpha : \alpha < \beta\} \rangle$ such that for all $\alpha < \beta$ either $f_\alpha \in G_\beta$ or $\langle G_\beta, f_\alpha \rangle$ is not cofinitary.

g_β will be constructed from finite approximations g_s, i.e. $g_\beta = \bigcup_{s \in \mathbb{N}} g_{\beta,s}$. W_{G_β} is a countable set; enumerate it as $\langle w_n : n \in \mathbb{N} \rangle$.

At sub-stage $s \in \mathbb{N}$ we have already constructed $g_{\beta,s}$ ($g_{\beta,0} := \emptyset$), and we will construct $g_{\beta,s+1}$ from $g_{\beta,s}$ in three steps:

- $g^1_{\beta,s} = g_{\beta,s} \cup \{(n,k)\}$, where n is the least number not in the domain of $g_{\beta,s}$ and k is the least number for which $g^1_{\beta,s}$ is a good extension of $g_{\beta,s}$ with respect to $w_0, \ldots, w_s$ (k exists by the domain extension lemma).

- $g^2_{\beta,s} = g^1_{\beta,s} \cup \{(n,k)\}$, where k is the least number not in the range of $g^1_{\beta,s}$ and n is the least number for which $g^2_{\beta,s}$ is a good extension of $g^1_{\beta,s}$ with respect to $w_0, \ldots, w_s$ (n exists by the range extension lemma).

- $g_{\beta,s+1} = g^2_{\beta,s}$ if $\langle G_\beta, f_\beta \rangle$ is not cofinitary, or $f_\beta \in G_\beta$; otherwise $g_{\beta,s+1} = g^2_{\beta,s} \cup \{(n, f(n))\}$, where n is the least number such that $g_{\beta,s+1}$ is a good extension of $g^2_{\beta,s}$ with respect to $w_0, \ldots, w_s$ (n exists by the hitting f lemma).

$\langle G_\beta, g_\beta\rangle$ will be cofinitary: for any $h \in \langle G_\beta, g_\beta\rangle$ there is a $w \in W_{G_\beta}$ such that $h = w(g_\beta)$. And for any $w \in W_{G_\beta}$ there is a stage t such that w and all of its subwords are included in $w_0, \ldots, w_t$, so Lemma III.7 applies.

The group $G := \langle\{g_\alpha : \alpha < \omega_1\}\rangle$ is maximal cofinitary: for any f_β either $f_\beta \in G_{\beta+1} \subseteq G$, or $\langle G_{\beta+1}, f_\beta\rangle$ is not cofinitary (if $\langle G_\beta, f_\beta\rangle$ is cofinitary and $f_\beta \notin G_\beta$, we infinitely often add a pair from f to g_β in the third step above), so also $\langle G, f_\beta\rangle$ is not cofinitary.

Note that G is a group that is freely generated by the g_α $(\alpha < \omega_1)$: the method of good extensions, without additional work, always leads to free groups. For any word w as soon as we take good extensions with respect to it and its subwords we do not add any fixed points we are not forced to add. And because at this stage we have only a finite approximation to the new generator, there are only finitely many of those.

3.2 Orbits

3.2.1 No Maximal Cofinitary Group Has Countably Many Orbits

This subsection is devoted to the proof of the following theorem. Note that we are only looking at orbits of the action of G on $\mathbb{N}$ obtained from the inclusion $G \subseteq \mathrm{Sym}(\mathbb{N})$.

Theorem III.11. *A maximal cofinitary group has only finitely many orbits.*

Suppose G is a cofinitary group with infinitely many orbits. Fix an enumeration without repetitions $\langle O_i : i \in \mathbb{N}\rangle$ of all orbits of G. From these data we define a function h such that $h \notin G$ and $\langle G, h\rangle$ is cofinitary, so G is not maximal. We will first define h, show some of its properties and finally show how these properties can be used to show that h is as required.

We define $h : \mathbb{N} \to \mathbb{N}$ by a sequence of finite approximations h_s, $s \in \mathbb{N}$. Set $h_0 := \emptyset$, and suppose h_s has been defined. Let $n := \min\big((\mathbb{N} \setminus \mathrm{dom}(h_s)) \cup (\mathbb{N} \setminus \mathrm{ran}(h_s))\big)$ and $m := \min O_j$, where j is the least number such that $O_j \cap \big(\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s)\big) = \emptyset$. Then set $h_{s+1} := h_s \cup \{\langle n, m\rangle\}$ if $n \notin \mathrm{dom}(h_s)$ and $h_{s+1} := h_s \cup \{\langle m, n\rangle\}$ otherwise.

Clearly $h \in \mathrm{Sym}(\mathbb{N}) \setminus G$; we only need to verify that for all $w(x) \in W_G$ the function $w(h)$ has finitely many fixed points, or is the identity. It will in fact be the case that all $w(h)$ (except the identity word) have only finitely many fixed points; showing this will take some work.

First note that for all O_i and O_j there is at most one pair $\langle a, b\rangle \in h$ such that $a \in O_i$ and $b \in O_j$. But in fact *much* more is true. This much more is described by the following definition, which also describes the picture from which this proof developed.

Definition III.12. The *G-orbits tree of h* has vertex set $\{O_j : j \in \mathbb{N}\}$. It has an edge between O_j and O_i if there is an $n \in O_j$ such that $h(n) \in O_i$.

We need to see that this in fact defines a tree. Suppose not, then there is a cycle $O_{n_0}, O_{n_1}, \ldots, O_{n_l} = O_{n_0}$ and for all $0 \le i < l$ vertex O_{n_i} is connected to vertex $O_{n_{i+1}}$. This means that for every $0 \le i < l$ there is a pair $\langle a, b\rangle \in h$ such that $a \in O_{n_i}$ and $b \in O_{n_{i+1}}$ or $a \in O_{n_{i+1}}$ and $b \in O_{n_i}$. By the observation above these pairs are unique. Let $s \in \mathbb{N}$ be the least s such that all pairs $\langle a, b\rangle$ used in this cycle are in h_{s+1}. Since s is least with this property the unique pair $\langle a, b\rangle \in h_{s+1} \setminus h_s$ is used in the cycle. Then $\langle a, b\rangle$ connects some O_{n_j} with one of its neighbors, $O_{n_{j-1}}$ or $O_{n_{j+1}}$. But each of these is already connected to its other neighbor, so $a \in O_k$ and $b \in O_l$ such that $O_k \cap (\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s)) \neq \emptyset$ and $O_l \cap (\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s)) \neq \emptyset$. This however means that the pair $\langle a, b\rangle$ does not satisfy the defining criterion for inclusion in h_{s+1}; so we have the contradiction we were looking for.

The next definition gives us a way to talk about the process of evaluating a word $w(h)$ on a number n. The orbit path defined here can be looked at as a walk on the vertices of the G-orbit tree of h.

Definition III.13. For $m \in \mathbb{N}$, $w(x) = g_0 x^{k_0} g_1 \cdots x^{k_{l-1}} g_l \in W_G$ and $h \in \mathrm{Sym}(\mathbb{N})$ we define the *orbit path* of n in $w(h)$ to be the sequence of orbits the evaluation passes through — that is $\bar{l} = \langle l_i : 0 \leq i \leq \mathrm{lh}(w) \rangle$ where $l_i = j$ iff $z_i \in O_j$ with $\bar{z}$ the evaluation path for w on n.

One of the essential features of the function h we have defined is that for any $n \in \mathbb{N}$ and $w(x) \in W_G$, the evaluation path for n and the orbit path of n determine each other. This equivalence will be useful as in a word with infinitely many fixed points the action on the orbits allows us to conclude that one of the g_i in $w(x)$ has infinitely many fixed points which will be the, at that time, desired conclusion.

We are now ready to finish the proof of Theorem III.11. Suppose, towards a contradiction, that there is a $w \in W_G$ such that $w(h)$ has infinitely many fixed points. We will show that each fixed point of $w(h)$ gives rise to a fixed point in some g_i appearing in w.

Let n be one of the fixed points of $w(h)$ and $\bar{l}$ its orbit path. Let $l_i \in \bar{l}$ be such that O_{l_i} is the first vertex realizing the maximal distance from O_{l_0} in the G-orbit tree of h. The orbit $O_{l_{i-1}}$ preceding O_{l_i} is closer to O_{l_0}, so $w_{(i)} = x$ or x^{-1} (application of any member of G will not change the orbit we are in) and $(w \restriction i)(n) \in O_{l_{i-1}}$ and $(w \restriction i+1)(n) \in O_{l_i}$ with $\langle (w \restriction i)(n), (w \restriction i+1)(n) \rangle \in h$ or $\langle (w \restriction i+1)(n), (w \restriction i)(n) \rangle \in h$.

Assume the former; the other case is analogous.

Since the G-orbit tree of h is a tree, $O_{l_{i-1}}$ and O_{l_i} are connected by an edge and $O_{l_{i-1}}$ is strictly closer to O_{l_0} than O_{l_i}, all the other neighbors of O_{l_i} are strictly closer to O_{I_0} than O_{I_i}. This means that the first vertex after O_{I_i} different from O_{I_i} has to be equal to $O_{I_{i-1}}$. But as $\langle (w \restriction i)(n), (w \restriction i+1)(n) \rangle$ is the only pair in h allowing direct passage between $O_{I_{i-1}}$ and O_{I_i} this means we have to apply h^{-1} with input $(w \restriction i+1)(n)$ to get back to $O_{I_{i-1}}$.

We have the following situation in the orbit path of n:

$$(w \restriction i)(n) \in O_{I_{i-1}} \xrightarrow{h} (w \restriction i+1)(n) \in O_{I_i} \to \cdots$$
$$\to (w \restriction i+i)(n) \in O_{I_i} \xrightarrow{h^{-1}} (w \restriction i)(n) \in O_{I_{i-1}}$$

Now, in between arriving at O_{I_i} and leaving O_{I_i} we obviously stay in the same orbit. This means that between arriving at O_{I_i} and leaving we can only apply members of G. By the shape of w we apply exactly one member g_j of G. And by the work above this member has to fix $(w \restriction i+1)(n)$.

We now know that every fixed point of $w(h)$ gives rise to a fixed point in some g_l appearing in w. There is a j such that infinitely many fixed points of $w(h)$ give rise to a fixed point in that g_j. No two such fixed points of $w(h)$ can be associated to the same fixed point of g_j as for different points the jth members of their respective evaluation paths are never equal. (Note that here we are considering (g_j, j) the group element together with an indication of where it occurs in the word. It is possible that one group element occurs more than once.) This shows that this g_j appearing in w has infinitely many fixed points, contradicting that it is a member of the cofinitary group G.

3.2.2 A Maximal Cofinitary Group With Finitely Many Infinite Orbits

In this subsection we prove the following theorem:

Theorem III.14. *The continuum hypothesis implies that for every* $n \in \mathbb{N} \setminus \{0\}$ *and* $m \in \mathbb{N}$ *there exists a maximal cofinitary group with exactly* n *infinite orbits and* *exactly m finite orbits.*

Proof. Let $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{N}$ be given. Choose a partition $F_0 \dot\cup F_1 \cdots F_{m-1} \dot\cup O_0 \dot\cup O_1 \cdots O_{n-1} = \mathbb{N}$ with F_i finite and O_j infinite; these will be the orbits.

We have to ensure that the group G we construct satisfies the following four conditions.

1. G is transitive on all F_i and O_j,

2. G respects all F_i and O_j,

3. G is cofinitary,

4. G is *maximal* cofinitary.

We will construct sequences of generators $\langle g_\alpha^{F_i} : \alpha < \omega_1 \rangle$ $(i < m)$ and $\langle g_\alpha^{O_i} : \alpha < \omega_1 \rangle$ $(i < n)$ such that $g_\alpha^{F_i} \in \mathrm{Sym}(F_i)$ and $g_\alpha^{O_i} \in \mathrm{Sym}(O_i)$. We then define G to be the group generated by $\langle g_\alpha : \alpha < \omega_1 \rangle$, where $g_\alpha = (\bigcup_{i<m} g_\alpha^{F_i}) \cup (\bigcup_{i<n} g_\alpha^{O_i})$.

To ensure condition 1 we choose the first member in each of the sequences to generate a transitive group.

Condition 2 is ensured by the way the elements of the group G are obtained from the sequences of generators we construct.

To ensure condition 3 we use the method of good extensions on the infinite orbits. Since then the groups on the infinite orbits are freely generated and cofinitary, the same will be true for G.

To ensure condition 4 we pick an enumeration $\langle f_\alpha : \alpha < \omega_1 \rangle$ of $\mathrm{Sym}(\mathbb{N})$ and ensure in step α of the construction (which will be recursive with ω_1 steps) that either $f_\alpha \in G$ or there is a $w \in W_G$ such that $w(f_\alpha)$ has infinitely many fixed points but is not the identity.

We now give the details of the construction. We recursively construct the sequences of generators on the infinite orbits, and in the first step fix the sequences on the finite orbits.

For each $i < m$ choose a permutation of F_i that generates a transitive group. Then set each $g_\alpha^{F_i}$ to be equal to that permutation. If we make sure that the permutations on the infinite orbits are cofinitary and generate the group freely, this takes care of the finite orbits (the group generated by the $g_\alpha^{F_i}$ clearly is transitive on F_i, and as the generators on the O_i generate the group freely we don't have to consider relations between different $g_\alpha^{F_i}$). So from now on we will restrict our attention to the infinite orbits.

For each $i < n$ choose a transitive permutation of O_i, and set $g_0^{O_i}$ equal to that permutation.

Note that the group generated by g_0 is free, cofinitary, and transitive when restricted to each of the sets F_i $(i < m)$ and O_i $(i < n)$.

At step $\beta < \omega_1$ we have constructed the sequences $\langle g_\alpha^{O_i} : \alpha < \beta \rangle$ $(i < n)$. And the $g_\alpha^{O_i}$ freely generate a cofinitary subgroup of $\mathrm{Sym}(O_i)$. Also for all $\alpha < \beta$ we have ensured that either $f_\alpha \in \langle g_\alpha : \alpha < \beta \rangle$ or there is a $w(x) \in W_{\langle \{g_\alpha : \alpha < \beta\} \rangle}$ such that $w(f_\alpha)$ has infinitely many fixed points, but is not the identity.

There are two cases.

Case 1: $f_\beta \cap (O_0 \times O_0)$ is infinite.

Then on in $\mathrm{Sym}(O_0)$ we perform the construction of $g_\beta^{O_0}$ using the domain, range, and hitting f lemma (which, as we remarked, also works for infinite partial functions), so construct a bijection such that $f \cap g^{O_0}$ is infinite, but not all of g^{O_0}. On the other infinite orbits, we only use domain and range extension to construct a new generator.

Case 2: Otherwise there is an $i \neq 0$ such that $f_\beta \cap (O_0 \times O_i)$ is infinite.

Let $R := \operatorname{ran}(f_\beta \cap (O_0 \times O_i))$. Then R is an infinite subset of O_i. We now use domain and range extension to construct $g_\beta^{O_i}$ such that $g_\beta^{O_i} \cap (R \times R)$ is infinite. This means that $f_\beta^{-1} g_\beta^{O_i} f_\beta$ is an infinite partial function $O_0 \rightharpoonup O_0$. And we can construct $g_\beta^{O_0}$ by using the domain extension, range extension, and hitting f lemmas, so that $g_\beta^{O_0} \cap f_\beta^{-1} g_\beta^{O_i} f_\beta$ is infinite, but not all of g^{O_0}. This shows there is a $w(x) \in W_{G_\beta}$ ($w(x) = g_\beta^{-1} x^{-1} g_\beta x$) such that $w(f_\beta)$ has infinitely many fixed points but is not the identity.

On the other infinite orbits, we only use domain and range extension to construct a new generator.

In both cases we get freely generated cofinitary groups, and in both cases we take care of the function f_β. Therefore we have constructed a maximal cofinitary group. □

3.3 Isomorphism Types

3.3.1 A Maximal Cofinitary Group Universal for Countable Groups

This subsection is dedicated to proving the following theorem.

Theorem III.15. *The continuum hypothesis implies that there exists a maximal cofinitary group G such that every countable group H embeds into G.*

Domain and Range Extension

Here we generalize the domain and range extension lemmas.

Definition III.16. (i). Let $G \leq \operatorname{Sym}(\mathbb{N})$ and let $\langle x_n : n \in \mathbb{N} \rangle$ be a sequence of variables. Then $W_{G,n}$ is $G * F(x_0, \ldots, x_n)$, the free product of G and the free group on the generators $x_0, \ldots, x_n$; i.e. $W_{G,n}$ is the set of words of the form $w = w(x_0, \ldots, x_n) = g_0 \bar{x}_0 g_1 \bar{x}_1 \cdots \bar{x}_k g_{k+1}$, where $g_i \in G$ for $0 \leq i \leq k+1$, $g_i \neq \mathrm{Id}$ for $1 \leq i \leq k$, and $\bar{x}_i$ is a nonempty reduced word in $F(x_0, \ldots, x_n)$.

(ii). For $\bar{p} = p_0, \ldots, p_n$, $\bar{q} = q_0, \ldots, q_n$ with all $p_i : \mathbb{N} \rightharpoonup \mathbb{N}$ finite injective, all $q_i : \mathbb{N} \rightharpoonup \mathbb{N}$ finite injective and $q_i \supseteq p_i$, we call $\bar{q}$ a *good* extension of $\bar{p}$ with respect to $w \in W_{G,n}$ if the following holds:

For all $l \in \mathbb{N}$ such that

$$w(\bar{q})(l) = l \text{ and } w(\bar{p})(l) \text{ is undefined}$$

there exists an $k \in \mathbb{N}$ and z, u subwords of w such that

$$(\dagger) \qquad \begin{gathered} w = u^{-1}zu \text{ without cancellation,} \\ z(\bar{p})(k) = k \text{ and } u(\bar{q})(l) = k. \end{gathered}$$

For $n = 0$ this reduces to the original definition of good extension.

(iii). If G is countable, then $W_{G,n}$ is countable; enumerate it by $\langle w_k : k \in \mathbb{N} \rangle$. Now we define a partial order $\mathbb{P}_{G,n} = \langle P, \leq \rangle$. P is the set of length $n+1$ sequences of finite injective functions $\mathbb{N} \to \mathbb{N}$, and $\bar{q} \leq \bar{p}$ if $\bar{q}$ is a good extension of $\bar{p}$ with respect to all words $\{w_k : k \leq |\bar{p}|\}$ and their subwords.

This order depends on the enumeration $\langle w_k : k \in \mathbb{N} \rangle$ of $W_{G,n}$, but in a way that will never matter to us.

We can now state and prove the versions of the domain and range extension lemmas we need.

Lemma III.17 (Domain Extension Lemma)**.** *Let G be a countable cofinitary group, $\bar{p} = p_0, \ldots, p_n$ finite injective functions, $i \leq n$, $a \in \mathbb{N} \setminus \operatorname{dom}(p_i)$ and $w \in W_{G,n}$ then for all but finitely many $b \in \mathbb{N}$ the sequence $p_0, \ldots, p_i \cup \{(a, b)\}, \ldots, p_n$ is a good extension of $\bar{p}$ with respect to w.*

Proof. Using the original domain and range extension lemmas we can in turn extend each $p_j, j \neq i$, to a permutation $\tilde{p_j}$ so that $\langle G, \{\tilde{p}_j | j \leq n, j \neq i\} \rangle$ is a cofinitary group, and the $\tilde{p}_j, j \leq n, j \neq i$, are free generators.

Then, by the original domain extension lemma, for all but finitely many $b \in \mathbb{N}$, $p_i \cup \{(a,b)\}$ is a good extension of p_i with respect to the word $w' = w(\tilde{p}_0, \ldots, \tilde{p}_{i-1}, x, \tilde{p}_{i+1}, \ldots, \tilde{p}_n)$.

We claim that for such b the sequence $p_0, \ldots, p_i \cup \{(a,b)\}, \ldots, p_n$ is a good extension of $\bar{p}$ with respect to w, which we see as follows: For any fixed point $s \in \mathbb{N}$ of $w(p_0, \ldots, p_i \cup \{(a,b)\}, \ldots, p_n)$ that is not a fixed point of $w(\bar{p})$ we have that s is a fixed point of $w(\tilde{p}_0, \ldots, p_i \cup \{(a,b)\}, \ldots, \tilde{p}_n)$ that is not a fixed point of $w(\tilde{p}_0, \ldots, p_i, \ldots, \tilde{p}_n)$. So as $p_i \cup \{(a,b)\}$ is a good extension of p_i with respect to w' we have subwords u, z of w' as in (†) (for words with only one variable). As the $\tilde{p}_m$ were free generators by replacing in them the $\tilde{p}_l$ by x_l we get $w = u^{-1}zu$ as in (†) (for words with n variables), showing this is a good extension. □

Lemma III.18 (Range Extension Lemma)**.** *Let G be a countable cofinitary group, $\bar{p} = p_0, \ldots, p_n$ finite injective functions, $i \leq n$, $l \in \mathbb{N} \setminus \operatorname{ran}(p_i)$ and $w \in W_{G,n}$ then for all but finitely many $k \in \mathbb{N}$ the sequence $p_0, \ldots, p_i \cup \{(k,l)\}, \ldots, p_n$ is a good extension of $\bar{p}$ with respect to w.*

Proof. From the domain extension lemma by using $\bar{p}^{-1} = p_0^{-1}, \ldots, p_n^{-1}$ and $\tilde{w} = w^{-1}$. □

Corollary III.19 (Domain and Range Extension Lemma)**.** *Let G be a countable cofinitary group. Then the sets $D_{i,k} := \{\bar{q} \in P : k \in \operatorname{dom}(q_i)\}$ and $R_{i,k} := \{\bar{q} \in P : k \in \operatorname{ran}(q_i)\}$, $i \leq n$ and $k \in \mathbb{N}$, are dense in $\mathbb{P}_{G,n}$.*

From this corollary it follows that we can add n permutations $\bar{f}$ at a time to a countable cofinitary group G in such a way that $\langle G, \bar{f} \rangle$ is cofinitary and the group generated by $\bar{f}$ is free.

Enumerate $\{D_{i,k} : i \leq n \text{ and } k \in \mathbb{N}\} \cup \{R_{i,k} : i \leq n \text{ and } k \in \mathbb{N}\}$ by $\langle D_s : s \in \mathbb{N}\rangle$. Then let $\bar{p}_0 = \emptyset$, and recursively choose $\bar{p}_{s+1} \leq \bar{p}_s$ such that $\bar{p}_{s+1} \in D_s$.

Clearly this gives that all $f_i := \bigcup_{s\in\mathbb{N}}(\bar{p}_s)_i$, where $(\bar{p}_s)_i$ is the i-th component of $\bar{p}_s$, are bijections. It remains to verify that the group $\langle G, \bar{f}\rangle$ is cofinitary. For this it is sufficient to show that for any $w \in W_{G,n}$ the permutation $w(\bar{f})$ is cofinitary. In fact it will only have finitely many fixed points.

Let $w \in W_{G,n}$; so $w = w_k$ for some $k \in \mathbb{N}$. $|\bar{p}_k| \geq k$ so that for $s \geq k$, $\bar{p}_{s+1}$ is a good extension of $\bar{p}_s$ with respect to w_k and its subwords. Suppose, towards a contradiction, that $w(\bar{f})$ has infinitely many fixed points. Let w' be the shortest subword of w such that $w'(\bar{f})$ has infinitely many fixed points.

However, w' being a subword of w, from stage k on we only take good extensions with respect to this word, and $w'(\bar{p}_k)$ has only finitely many fixed points (as it is a finite permutation). So for every stage $s > k$, when $w'(\bar{p}_s)$ has a fixed point that $w'(\bar{p}_{s-1})$ does not have, we find a subword w'' of w' and a fixed point for $w''(\bar{p}_{s-1})$ (since $\bar{p}_s$ is a good extension). Every time we find a different fixed point for some subword w'' of w'. Since there are only finitely many subwords w'', one of them has infinitely many fixed points, contradicting the assumption that w' was the shortest subword with infinitely many fixed points.

Finitely Generated Groups

Here we'll show how to add an isomorphic copy of any finitely generated group to any countable cofinitary group.

So let H be a finitely generated group, with generators $h_0, \ldots, h_n$. Let $x_0, \ldots, x_n$ be variables, where obviously the intention is that x_i will represent h_i, and we write $\bar{x}$ for the sequence of x's. Let $W_{H,\mathrm{Id}}$ be the set of words $w(\bar{x}) \in F(\bar{x})$ such that $w(\bar{h}) := w(h_0, \ldots, h_n) = \mathrm{Id}$, i.e. the set of words representing the identity (which is a normal subgroup of $F(\bar{x})$). It follows that H is isomorphic to $F(\bar{x})/W_{H,\mathrm{Id}}$.

To make H act cofinitarily we plan to construct permutations of $\mathbb{N}$ $f_0, \ldots, f_n$ such that

- for all $w \in W_{H,\mathrm{Id}}$, $w(\bar{f}) = \mathrm{Id}$, and
- for all $w \in F(\bar{x}) \setminus W_{H,\mathrm{Id}}$, $w(\bar{f})$ has finitely many fixed points.

If we do this we clearly get the following theorem:

Theorem III.20. *Any finitely generated group has a cofinitary action.*

But we want to start with G a countable cofinitary group and add $\bar{f}$ such that

- the group generated by $\bar{f}$ is isomorphic to H,
- $\langle G, \bar{f} \rangle$ is cofinitary.

So let G be a countable cofinitary group, and set $W_{G,H} := G * (F(\bar{x})/W_{H,\mathrm{Id}})$. This is a cou980Table set, so we can enumerate it by $\langle w_\mu x$

Note that for $w_1, w_2 \in W_{G,n}$, if $w_1(\bar{h}) = w_2(\bar{h})$ then these words represent the same group element of $W_{G,H}$. Also note that the notion of (G,H)-good extension depends on which generators for H we have chosen and which enumeration of $W_{G,H}$. But this dependence will never matter to us. Lastly, if $\bar{q}$ is a good extension of $\bar{p}$ with respect to all words w_i, $i \leq |\bar{p}|$, and all their subwords, then $\bar{q}$ is a (G,H)-good extension of $\bar{p}$.

We define a poset $\mathbb{P}_{G,H} = \langle P, \leq \rangle$ associated with the groups G and H. The elements of P are $\bar{p} = (p_0, \ldots, p_n)$, finite sequences of length $n+1$ of finite injective partial maps $\mathbb{N} \to \mathbb{N}$, such that for all $w \in W_{H,\mathrm{Id}}$ we have $w(\bar{p}) \cong \mathrm{Id}$ ($w(\bar{p})$ is the identity where defined). $\bar{q} \leq \bar{p}$ if $\bar{q}$ is a (G,H)-good extension of $\bar{p}$.

Note that if $\bar{q}$ is a good extension of $\bar{p}$ with respect to all words w_i, $i \leq |\bar{p}|$, and their subwords, then not necessarily $\bar{q} \in P$.

If $\bar{p} \in P$ we say $\bar{q}$, a finite sequence of length $n+1$ of injective partial maps $\mathbb{N} \to \mathbb{N}$, is obtained from $\bar{p}$ by *applying relations* if

$$(a,b) \in q_i \quad \Leftrightarrow \quad \exists w'[x_i w' \in W_{H,\mathrm{Id}} \wedge w'(\bar{p})(b) = a].$$

Lemma III.21. *If $\bar{q}$ is obtained from $\bar{p}$ by applying relations, then*

(i). for all $i \leq n$, $p_i \subseteq q_i$;

(ii). applying relations to $\bar{q}$ doesn't add anything;

(iii). $\bar{q} \in P$;

(iv). $\bar{q} \leq \bar{p}$

Proof. (i) is immediate since $x_i x_i^{-1} \in W_{H,\mathrm{Id}}$.

For (ii): if $x_j w \in W_{H,\mathrm{Id}}$ and $w(\bar{q})(b) = a$, then there exists a w' such that $(x_j w)^{-1} x_j w' \in W_{H,\mathrm{Id}}$ and $w'(\bar{p})(b) = a$.

For (iii): if $w \in W_{H,\mathrm{Id}}$ is such that $w(\bar{q}) \ncong \mathrm{Id}$, then there is a word w' such that $w^{-1}w' \in W_{H,\mathrm{Id}}$ such that $w'(\bar{p}) \ncong \mathrm{Id}$. Also $\bar{q}$ is finite since any map in this sequence only has pairs (a, b) in it where $a, b \in \bigcup_{i \leq n} (\mathrm{dom}(p_i) \cup \mathrm{ran}(p_i))$.

For (iv): if $w \in W_{G,H}$ is such that $w(\bar{q})(n)$ is defined, then there is a word $w' \in W_{G,H}$ such that $w'(\bar{h}) = w(\bar{h})$ and $w'(\bar{p})(n)$ is defined and has the same value. □

Now we define for $i \leq n$ and $k \in \mathbb{N}$ the sets $D_{i,k} := \{\bar{p} \in P : k \in \mathrm{dom}(p_i)\}$ and $R_{i,k} := \{\bar{p} \in P : k \in \mathrm{ran}(p_i)\}$.

Lemma III.22. *For all $i \leq n$ and $k \in \mathbb{N}$, the sets $D_{i,k}$ and $R_{i,k}$ are dense.*

Proof. The proofs for $D_{i,k}$ and $R_{i,k}$ are basically the same, so we'll only show that $D_{i,k}$ is dense for some $i \leq n$ and $k \in \mathbb{N}$. (Also that $R_{i,k}$ is dense follows from $D_{i,k}$ being dense for any group and set of generators; just consider the group generated by the inverses of the generators.)

So let $i \leq n$, $k \in \mathbb{N}$ and $\bar{p} \in P$. We need to find $\bar{r} \in P$ s.t. $\bar{r} \leq \bar{p}$ and $\bar{r} \in D_{i,k}$.

First apply relations to $\bar{p}$ to get $\bar{q}$. If $\bar{q} \in D_{i,k}$ we are done by (iii) and (iv) of the previous lemma, so suppose $\bar{q}$ is not in $D_{i,k}$.

Now by the domain extension lemma, for all but finitely many $l \in \mathbb{N}$, $(q_0, \ldots, q_i \cup \{(k,l)\}, \ldots, q_n)$ is a good extension of $\bar{q}$ with respect to all words w_i, $i \leq |\bar{p}|$, and their subwords. Choose such l such that $l > \max\{\{k\} \cup \bigcup_{i \leq n} (\mathrm{dom}(q_i) \cup \mathrm{ran}(q_i))\}$.

Let $\bar{r} = (q_0, \ldots, q_i \cup \{(k,l)\}, \ldots, q_n)$ for this l. If $\bar{r} \in P$ then clearly $\bar{r} \leq \bar{p}$ since a good extension with respect to all words $\{w_i : i \leq |\bar{p}|\}$ and their subwords is a (G,H)-good extension. So, towards a contradiction, suppose that there is a $w \in W_{H,\mathrm{Id}}$ such that $w(\bar{r}) \ncong \mathrm{Id}$ and let w be the shortest such word. Since $w(\bar{q}) \cong \mathrm{Id}$ (since $\bar{q} \in P$ by (iii) of the previous lemma), this new computation $w(\bar{r})(a) \neq a$ uses the pair (k,l). Since $l > \max\{\bigcup_{i \leq n} (\mathrm{dom}(q_i) \cup \mathrm{ran}(q_i))\}$ this pair has to be used at the beginning or the end; if it is used in a location in the middle then it needs to be used again immediately (in the opposite direction). This would mean w has a subword $x_i^{-1}x_i$ or $x_ix_i^{-1}$ contradicting its minimality. If (k,l) is used at the beginning and the end then $w = x_iw'x_i^{-1}$, $l = a$ and $w(\bar{r})(a) = a$ contrary to the assumption that a is not a fixed point. So (k,l) is used only once either at the beginning of the word or at the end of the word. This however contradicts $\bar{q}$ having been obtained by applying relations, if there is a word w such that $w = w'x_i$ and $w'(\bar{q})(k)$ is defined, then $(k, w'(\bar{q})(k)) \in q_i$ (for exact correspondence with the definition of applying relations consider $x_i(w')^{-1}$), showing k was already in the domain of q_i which means $\bar{q} \in D_{i,k}$. □

From this lemma we get the following theorem by reasoning similar to that at the end of the previous subsection (page 59).

Theorem III.23. *For G a countable cofinitary group and H a finitely generated group, there exist $\bar{f}$ such that the group generated by $\bar{f}$ is isomorphic to H and $\langle G, \bar{f}\rangle$ is cofinitary.*

Countable Groups

Here we'll adapt the method from the previous section to any countable group H (as opposed to only finitely generated).

So let H be a countable group, with generators $\vec{h} := \langle h_n : n \in \mathbb{N}\rangle$. Let $\vec{x} := \langle x_n : n \in \mathbb{N}\rangle$ be variables, where obviously the intention is that x_n will represent h_n. Let $W_{H,\mathrm{Id}}$ be the set of words $w(\vec{x}) \in F(\vec{x})$ such that $w(\vec{h}) = \mathrm{Id}$; i.e. the set of words representing the identity. With this we have as before that H is isomorphic to $F(\vec{x})/W_{H,\mathrm{Id}}$.

We want as before to start with G a countable cofinitary group and add $\vec{f}$ such that

- the group generated by $\vec{f}$ is isomorphic to H,
- $\langle G, \vec{f} \rangle$ is cofinitary.

So let G be a countable cofinitary group, and set $W_{G,H} := G * (F(\vec{x})/W_{H,\mathrm{Id}})$. This is a countable set, so we can enumerate it by $\langle w_n : n \in \mathbb{N} \rangle$.

If $\vec{p} := \langle p_n : n \in \mathbb{N} \rangle$ is a sequence of finite injective functions $\mathbb{N} \rightharpoonup \mathbb{N}$ we define its support $\mathrm{supp}(\vec{p}) := \{i \in \mathbb{N} : p_i \neq \emptyset\}$.

The notion of (G, H)-good extension is the same as before except that $\vec{p}$ and $\vec{q}$ are sequence of length ω of finite injective functions, and in stead of finite sequences $\bar{y}$ we use infinite sequences $\vec{y}$ (where y is x, p, q, or h).

The same remarks apply to this order as in the last section.

We define a poset $\mathbb{P}_{G,H} = \langle P, \leq \rangle$ associated with the groups G and H. The only difference in the definition with the poset from the last section is that its elements are length ω sequences with finite support of finite injective partial maps.

Now we get to the major difference with the work before. If we apply all relations to an element of the poset $\mathbb{P}_{G,H}$, then we are likely to get a sequence with infinite support. To avoid this we "localize" the applying of relations as follows.

If $\vec{p} \in P$ and A is a finite set of natural numbers, we say $\vec{q}$, a sequence of injective partial maps $\mathbb{N} \to \mathbb{N}$, is obtained from $\vec{p}$ by A-*applying relations* if

$$(a, b) \in q_i \quad \Leftrightarrow \quad (\exists w' \, [x_i w' \in W_{H,\mathrm{Id}} \wedge w'(\vec{p})(b) = a]) \wedge (p_i \neq \emptyset \text{ or } i \in A)\,.$$

Lemma III.21 where we replace applying relations by A-applying relations can be proved as before.

Now we define for $i, k \in \mathbb{N}$ the sets $D_{i,k} := \{\vec{p} \in \mathbb{P}_{G,H} : k \in \mathrm{dom}(p_i)\}$ and $R_{i,k} := \{\vec{p} \in \mathbb{P}_{G,H} : k \in \mathrm{ran}(p_i)\}$.

Lemma III.24. *For all $i \leq n$ and $k \in \mathbb{N}$, the sets $D_{i,k}$ and $R_{i,k}$ are dense.*

Proof. The proof is basically the same as the proof of Lemma III.22 except that we don't apply relations, but we $\{i\}$-apply relations. □

From this lemma we get the following theorem by reasoning similar to previous section.

Theorem III.25. *For G a countable cofinitary group and H a countable group, there exist $\vec{f}$ such that the group generated by $\vec{f}$ is isomorphic to H and $\langle G, \vec{f}\rangle$ is cofinitary.*

From this we get the following theorem

Theorem III.26. *The Continuum Hypothesis implies that there exists a maximal cofinitary group G such that every countable group H embeds into G.*

Proof. Enumerate all countable groups $\langle H_\alpha : \alpha < \omega_1\rangle$ and all permutations of $\mathbb{N}$ $\langle h_\alpha : \alpha < \omega_1\rangle$. At stage α we have already constructed the group G_α such that for all $\gamma < \alpha$ the group H_γ embeds into G_α and $h_\gamma \in G_\alpha$ or $\langle G_\alpha, h_\gamma\rangle$ is not cofinitary. Then extend the group G_α by H_α (by applying Theorem III.25 with H equal to H_α and G equal to G_α) obtaining the group G'_α. When $h_\alpha \notin G'_\alpha$ and $\langle G'_\alpha, h_\alpha\rangle$ is cofinitary, add a function hitting h_α infinitely often (this is done by using the domain and range extension lemmas, and the hitting f lemma). □

It should be noted that this is not the first proof of Theorem III.25: in [10] we found that Truss and Adeleke had proved the following theorem:

Theorem III.27. *Let G and H be countable cofinitary groups. Then there is a permutation $f : \mathbb{N} \to \mathbb{N}$ such that $\langle G, fHf^{-1}\rangle$ is cofinitary.*

3.3.2 Forcing Cofinitary Actions

In this subsection we start our investigation of consistency of the existence of cofinitary actions of groups for which we can't outright find one. We'll focus on the group $H := \bigoplus_{\alpha<\aleph_1} \mathbb{Z}_2$ (as this is one we understand already, and the general case is not yet quite clear).

Let $\vec{e} := \langle e_\alpha : \alpha < \aleph_1 \rangle$ be the standard generating set for H ($e_\alpha(\alpha) = 1$ and $e_\alpha(\beta) = 0$ for all $\beta \neq \alpha$).

Let $\vec{x} := \langle x_\alpha : \alpha < \aleph_1 \rangle$ be a sequence of variables, and $W_{H,\mathrm{Id}}$ the set of words $w(\vec{x})$ with $w(\vec{e}) = \mathrm{Id}$. Then $F(\vec{x})/W_{H,\mathrm{Id}}$ is isomorphic to H. Also note that for this group $W_{H,\mathrm{Id}}$ is generated by $\{x_\alpha^2 : \alpha < \aleph_1\} \cup \{[x_\alpha, x_\beta] : \alpha, \beta < \aleph_1\}$ (where $[x_\alpha, x_\beta] = x_\alpha^{-1} x_\beta^{-1} x_\alpha x_\beta$, the commutator of x_α and x_β).

We define a poset $\mathbb{P}_H = \langle P, \leq \rangle$. The set P consists of those sequences $\vec{p}$ of length $\aleph_1$ of partial finite functions $N \rightharpoonup \mathbb{N}$ with finite support (i.e. $\{\alpha < \omega_1 : (\vec{p})_\alpha \neq \emptyset\}$ is finite) for which $w(\vec{p}) \cong \mathrm{Id}$ for all $w \in W_{H,\mathrm{Id}}$. For I a finite subset of $\aleph_1$ we write x_I for $\{x_\alpha : \alpha \in I\}$. Then we choose for any finite set $I \subseteq \aleph_1$ an enumeration $\langle w_n^I : n \in \mathbb{N} \rangle$ of $F(x_I)$. If $\vec{p} \in P$, then W_p is the set of words $\bigcup_{I \subseteq \mathrm{supp}(p)} \{w_n^I : n \leq |\vec{p}|\}$ and all their subwords. If $\vec{p}, \vec{q} \in P$, then $\vec{q} \leq \vec{p}$ iff for any word w in W_p if $w(\vec{q})(l) = l$ for some $l \in \mathbb{N}$, then there are u_1, u_2, z subwords of w, $m \in \mathbb{N}$ and $z' \in F(\vec{x})$ such that

- $w = u_1^{-1} z u_2$ without cancellation,
- $u_1(\vec{e}) = u_2(\vec{e})$,
- $z(\vec{e}) = z'(\vec{e})$,
- $z'(\vec{p})(m) = m$, and
- $u_2(\vec{q})(l) = m$.

We first prove the following lemma.

Lemma III.28. $\mathbb{P}_H$ *is c.c.c..*

Proof. Let A be an uncountable set of elements from P.

Apply the Δ-system lemma to $\{\text{supp}(\vec{p}) : \vec{p} \in A\}$ to get A', an uncountable subset of A, such that $\{\text{supp}(\vec{p}) : \vec{p} \in A'\}$ is a Δ-system with root $r_{\aleph_1}$.

Then apply the Δ-system lemma to A' to get $\tilde{A}$, an uncountable subset of A', a Δ-system with root $\vec{r}$. Note that $\text{supp}(\vec{r}) \subseteq r_{\aleph_1}$.

In the coordinates in $r_{\aleph_1}$ there are only countably many possible finite extensions of $\vec{r}$, remove countably many members from $\tilde{A}$ so that for all the remaining members we have that $\vec{p} \restriction r_{\aleph_1} = \vec{r}$.

Let $C = \{\vec{p} : \vec{p}$ a sequence of length ω of finite partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$ with finite support$\}$. Any $\vec{p} \in P$ has finite support $I = \{\gamma_0 < \gamma_1 < \cdots < \gamma_k\}$, so to $\vec{p}$ we can assign $F(\vec{p}) = \langle p_{\gamma_0}, p_{\gamma_1}, \ldots, p_{\gamma_k}, \emptyset, \ldots\rangle$. Then $F : P \to C$. Since C is a countable set, there is an uncountable subset $\bar{A}$ of $\tilde{A}$ such that all members of $\bar{A}$ map to the same element.

We claim that all members of $\bar{A}$ are compatible. So let $\vec{p}, \vec{q} \in \bar{A}$, and let $\vec{s} = \vec{p} \cup \vec{q} = \langle p_\alpha \cup q_\alpha : \alpha < \aleph_1\rangle$. We get $\vec{s} \leq \vec{p}$ since $\vec{s}$ does not extend any $p_\alpha \neq \emptyset$, and W_p only contains words in variables x_α where $p_\alpha \neq \emptyset$. By the same reasoning $\vec{s} \leq \vec{q}$. Also all members of $\vec{s}$ are of order two, so satisfy the relations $\{x_\alpha^2 = \text{Id} : \alpha < \aleph_1\}$. If we can show that $\vec{s}$ also satisfies all $\{[x_\alpha, x_\beta] = \text{Id} : \alpha, \beta < \aleph_1\}$ it follows that $\vec{s} \in P$. Now suppose there are $\alpha, \beta \in \text{supp}(\vec{s})$ s.t. $[x_\alpha, x_\beta](\vec{s}) \ncong \text{Id}$. Then s_α and s_β are different, and one is from $\vec{p}$ the other from $\vec{q}$. W.l.o.g. $s_\alpha = p_\alpha$ and $s_\beta = q_\beta$. Since all elements of $\bar{A}$ map, under F, to the same element, there is a $\alpha' \in \text{supp}(\vec{p})$ s.t. $q_\beta = p_{\alpha'}$. But now we see that $[x_\alpha, x_\beta](\vec{s}) = [x_\alpha, x_{\alpha'}](\vec{p}) \cong \text{Id}$, a contradiction. So $\vec{s}$ satisfies all relations after all, and $\vec{s} \in P$ showing the compatibility of $\vec{p}$ and $\vec{q}$. $\square$

It is clear from our forcing and the reasoning in the last sections that a $\mathbb{P}_H$ generic set will give a cofinitary action for $\check{H}$. It remains to see that $\check{H} = H$ in the forcing extension. But since the forcing is c.c.c. no cardinals get collapsed, and if we take for H the group with underlying set the finite subsets of $\aleph_1$ with the obvious group operation, we get that $\check{H} = H$, showing the following theorem.

Theorem III.29. *There is a c.c.c. notion of forcing $\mathbb{P}_H$ such that $V^{\mathbb{P}_H} \models \bigoplus_{\alpha<\aleph_1} \mathbb{Z}_2$ has a cofinitary action.*

From this we get the following theorem.

Theorem III.30. *Martin's Axiom together with the negation of the continuum hypothesis implies $\bigoplus_{\alpha<\aleph_1} \mathbb{Z}_2$ has a cofinitary action.*

Proof. We don't need a generic set for all dense sets, only for the $\aleph_1$ many $D_{\alpha,i}$ and $R_{\alpha,i}$. □

3.4 Cardinal Characteristics

3.4.1 Diamond Results

In [29], Justin Moore, Michael Hrušák and Mirna Džamonja introduced weakenings of the diamond principle related to cardinal characteristics. We'll first study the effect of one of these weakenings of the diamond principle on families related to the symmetric group of the natural numbers.

Definition III.31. (i). $=^\infty$ is the relation on Baire space, ${}^{\mathbb{N}}\mathbb{N}$, of *infinite equality*: for $f, g \in {}^{\mathbb{N}}\mathbb{N}$, we have $f =^\infty g$ iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is infinite.

(ii). A function $F : {}^{<\omega_1}2 \to {}^{\mathbb{N}}\mathbb{N}$ is a Borel function if for all $\delta < \omega_1$ the function $F \restriction {}^{\delta}2 : {}^{\delta}2 \to {}^{\mathbb{N}}\mathbb{N}$ is Borel.

(iii). $\diamondsuit({}^{\mathbb{N}}\mathbb{N}, =^\infty)$ is the following guessing principle:

for every Borel function $F : {}^{<\omega_1}2 \to {}^{\mathbb{N}}\mathbb{N}$, there is a function $G : \omega_1 \to {}^{\mathbb{N}}\mathbb{N}$ such that for every $f : \omega_1 \to 2$ the set

$$\{\delta < \omega_1 : F(f \restriction \delta) =^\infty G(\delta)\}$$

is stationary.

A G related to F in this way is called a $\diamondsuit({}^{\mathbb{N}}\mathbb{N}, =^\infty)$-sequence for F.

We will study the effect of this $\diamondsuit$-principle (which is weaker than ordinary $\diamondsuit$) on the cardinal invariants $\mathfrak{a}_p$ and $\mathfrak{a}_g$ (see Definition I.32).

Theorem III.32. $\diamondsuit({}^{\mathbb{N}}\mathbb{N}, =^\infty)$ *implies* $\mathfrak{a}_p = \aleph_1$.

Proof. We will define a map F such that the $\diamondsuit({}^{\mathbb{N}}\mathbb{N}, =^\infty)$-sequence for it will help us build a sequence of permutations $\langle p_\alpha : \alpha < \omega_1\rangle$ which will be a maximal almost disjoint family of permutations of $\mathbb{N}$.

Let $C \subseteq \omega_1$ be the set of ordinals $\delta \in \omega_1$ such that $\delta = \omega + \omega \cdot \delta$. C is closed and unbounded in ω_1. Also fix a bijection $\theta : {}^{\mathbb{N}}2 \to \mathrm{Sym}(\mathbb{N})$, that is Borel. Let P_δ be the set $\{(\langle p_\alpha : \alpha < \delta\rangle, p) : \{p_\alpha : \alpha < \delta\} \cup \{p\}$ is a set of permutations$\}$. Pick $\delta \in C$ and let $\pi : \omega \mathbin{\dot\cup} \delta \times \omega \to \delta$ be an order isomorphism (where the order on $\omega \mathbin{\dot\cup} \delta \times \omega$ is such that all elements of ω are less then all elements of $\delta \times \omega$, and $\delta \times \omega$ is ordered lexicographically). Then for $x \in {}^{\delta}2$, define $D(x) = (\langle \theta(x \restriction \pi[\{\alpha\} \cdot \omega]), \theta(x \restriction \pi[\omega])\rangle$. D is a bijection ${}^{\delta}2 \to P_\delta$.

To define $F : {}^{<\omega_1}2 \to {}^{\mathbb{N}}\mathbb{N}$, by coding we let its range be ${}^{\mathbb{N}}(\mathbb{N} \cup {}^{<\omega}(\mathbb{N} \times \mathbb{N}))$. For $\delta \notin C$ we let $F \restriction {}^{\delta}2$ be any constant map; for $\delta \in C$ by the coding in the previous paragraph we let its domain be the set of pairs $(\langle p_\alpha : \alpha < \delta\rangle, p)$ with $\{p_\alpha : \alpha < \delta\} \cup \{p\}$ a family of permutations and define F as follows.

Fix for every $\delta < \omega_1$ a bijection $e_\delta : \mathbb{N} \to \delta$.

If $\{p_\alpha : \alpha < \delta\} \cup \{p\}$ is not almost disjoint, we define $F(\langle p_\alpha : \alpha < \delta\rangle, p)(n) = n$. Otherwise, we define $F(\langle p_\alpha : \alpha < \delta\rangle, p)(n)$ to be $\big((k_0, p(k_0)), (k_1, p(k_1)), \ldots,$

$(k_{6n}, p(k_{6n}))\big)$ with

- k_0 the least number such that $p(k_0) \notin \{p_{e_\delta(j)}(k_0) : j \leq n\}$, and
- k_{i+1} the least number strictly bigger than k_i with $p(k_{i+1}) \notin \{p_{e_\delta(j)}(k_{i+1}) : j \leq n\}$.

Since the family is almost disjoint, these k_i exist.

For any $\delta \in C$ the function as defined above restricted to those $(\langle p_\alpha : \alpha < \delta \rangle, p)$ for which $\{p_\alpha : \alpha < \delta\} \cup \{p\}$ is an almost disjoint family is continuous, and the set of $(\langle p_\alpha : \alpha < \delta \rangle, p)$ for which $\{p_\alpha : \alpha < \delta\} \cup \{p\}$ is an almost disjoint family is a Borel set. Composing this with the Borel coding D this shows that F is a Borel function.

Let $G : \omega_1 \to {}^{\mathbb{N}}\mathbb{N}$ be a $\Diamond({}^{\mathbb{N}}\mathbb{N}, =^\infty)$-sequence for this F. We define $G(\delta)(n)$ to be a *valid guess* for $\langle p_\alpha : \alpha < \delta \rangle$, a family of almost disjoint permutations, iff

- $G(\delta)(n) = \big((k_0, o_0), (k_1, o_1), \ldots, (k_{6n}, o_{6n})\big)$ for some $k_i, o_i \in \mathbb{N}$,
- all k_i are distinct, and
- all o_i are distinct and $o_i \notin \{p_{e_\delta(j)}(k_i) : j \leq n\}$.

Note that for any $\delta < \omega_1$, $n \in \mathbb{N}$, and any permutation p almost disjoint from all p_α, if $F(\langle p_\alpha : \alpha < \delta \rangle, p)(n) = G(\delta)(n)$ then $G(\delta)(n)$ is a valid guess for $\langle p_\alpha : \alpha < \delta \rangle$.

Now we use G to construct $\langle p_\alpha : \alpha < \omega_1 \rangle$ recursively. Suppose $\langle p_\alpha : \alpha < \delta \rangle$ has been defined. Then define p_δ recursively, $p_\delta := \bigcup_{s \in \mathbb{N}} p_{\delta,s}$, where

P1. $p_{\delta,0} := \emptyset$,

P2. $p'_{\delta,s+1} := p_{\delta,s}$ if $G(\delta)(s)$ is not a valid guess for $\langle p_\alpha : \alpha < \delta \rangle$,

P3. $p'_{\delta,s+1} := p_{\delta,s} \cup \{(k_i, o_i)\}$ if $G(\delta)(s) = \big((k_0, o_0), (k_1, o_1), \ldots, (k_{6s}, o_{6s})\big)$ is a valid guess for $\langle p_\alpha : \alpha < \delta \rangle$ and i is least such that $k_i \notin \mathrm{dom}(p_{\delta,s})$ and $o_i \notin \mathrm{ran}(p_{\delta,s})$,

P4. $p''_{\delta,s+1} := p'_{\delta,s+1} \cup \{(a, b)\}$ where a is the least number not in $\mathrm{dom}(p'_{\delta,s+1})$ and b is the least number not in $\mathrm{ran}(p'_{\delta,s+1})$ and not in $\{p_{e_\delta(j)}(a) : j \leq s\}$, and

P5. $p_{\delta,s+1} := p''_{\delta,s+1} \cup \{(c,d)\}$ where d is the least number not in $\operatorname{ran}(p''_{\delta,s+1})$ and c is the least number not in $\operatorname{dom}(p''_{\delta,s+1})$ and not in $\{p^{-1}_{e_\delta(j)}(d) : j \leq s\}$.

Note that $|p_{\delta,s}|$ is at most $3s$. This means we can do step P3, as the requirement $k_i \notin \operatorname{dom}(p_{\delta,s})$ excludes at most $3s$ pairs in $G(\delta)(s)$, $o_i \notin \operatorname{ran}(p_{\delta,s})$ excludes at most another $3s$ pairs in $G(\delta)(s)$, and $G(\delta)(s)$ has $6s+1$ pairs, always leaving at least one pair.

Now p_δ is a permutation (by P4 and P5) almost disjoint from all p_α, $\alpha < \delta$. This completes the construction of $\langle p_\alpha : \alpha < \omega_1 \rangle$.

It remains to see that this almost disjoint family of permutations is maximal. We do this by contradiction; suppose, therefore, that there is a permutation p almost disjoint from all p_α, $\alpha < \omega_1$. Then the set

$$\{\delta < \omega_1 : F(\langle p_\alpha : \alpha < \delta \rangle, p) =^\infty G(\delta)\}$$

is stationary. Remember that we use a coding for the inputs of the function F, and note that the sequence $\delta \mapsto (\langle p_\alpha : \alpha < \delta \rangle, p)$ determines a path $f : \omega_1 \to 2$ in the tree ${}^{<\omega_1}2$.

Now let $\delta \in C$ be a member of this set. Then $F(\langle p_\alpha : \alpha < \delta \rangle, p) =^\infty G(\delta)$, which means there are infinitely many n such that $G(\delta)(n)$ is a valid guess for $\langle p_\alpha : \alpha < \delta \rangle$, and all the pairs in $G(\delta)(n)$ belong to p. So we hit p infinitely often with p_δ (by P3), which is a contradiction. □

We now start to work towards the second theorem of this subsection; this will be a theorem similar to the above but for $\mathfrak{a}_g$.

Definition III.33. For $w \in W_G$ (see Definition III.2) and $p \subseteq q$ we call q a *very good extension* of p with respect to w if $w(q)$ has no more fixed points than $w(p)$.

Note that a very good extension is a good extension (see Definition III.6).

The following two lemmas show that we can construct a function F similar to the F in the proof of Theorem III.32 but for maximal cofinitary groups.

Lemma III.34. *Let H be a cofinitary group, $f \in \mathrm{Sym}(\mathbb{N}) \setminus H$ such that $\langle H, f \rangle$ is a cofinitary group and $w \in W_H$. Then for every $k \in \mathbb{N}$ there exists a finite set S of pairs from f such that for every finite injective map p with $|p|$ less than k there exists a pair (a, b) in S such that $p \cup \{(a, b)\}$ is a very good extension of p with respect to w.*

Proof. First we will find an infinite subset f' of f such that $w(f')$ has no fixed points, then we'll show that a big enough finite subset of f' exists. The first step ensures that we don't have to worry about fixed points caused by pairs from f alone. The second part is done by counting how many pairs from f' could combine with pairs from p to cause a fixed point.

Obtaining f' from f is done differently depending on whether $w(f)$ is the identity or not.

If $w(f)$ is not the identity, then it has only finitely many fixed points. Let f' be equal to f with for each of those finitely many fixed points one pair from f used in the evaluation path of that fixed point removed. We have ensured that $w(f')$ has no fixed points.

If $w(f)$ is the identity, then we know there is more than one occurrence of x in $w(x)$ (since $f \notin H$). So either there is an occurrence of x^2 or x^{-2}, or there is a subword of the form $x^{\,0}gx^{\,1}$, with $\epsilon_i \in \{-1, +1\}$ and $g \in H$. In either case there are only finitely many evaluation paths of $w(f)$ that use the same pair from f in both these selected occurrences of x (use that f has only finitely many fixed points for the first case, and that $f \notin H$ for the second case). Remove these finitely many pairs from f to obtain f''.

Now we have to find an infinite subset f' of f'' such that $w(f')$ is nowhere defined (which in this case is equivalent to not having fixed points).

We do this by recursively defining an enumeration $\{e_n : n \in \mathbb{N}\}$ of f'. Let $\prec$ be a wellorder of $\mathbb{N} \times \mathbb{N}$. Then define e_0 to be the $\prec$-least pair (a, b) in f'', and e_{n+1} to be the $\prec$-least pair in f'' that is not used in any evaluation path where a pair in $\{e_0, \ldots, e_n\}$ is also used.

We end up with an infinite f' such that $w(f')$ is indeed nowhere defined.

Now we examine for a given k and p, an injective map with $|p| = l \leq k$, how many pairs (a, b) of f' can have that $p \cup \{(a, b)\}$ is not a very good extension of p for w.

First there are at most $2l$ pairs (a, b) from f' that have $a \in \mathrm{dom}(p)$ or $b \in \mathrm{ran}(p)$. Remove these from f' to obtain $\tilde{f}$. Now we look at $w(p \cup \tilde{f})$; any fixed point of $w(p \cup \tilde{f})$ that was not a fixed point of $w(p)$ has an evaluation path where both pairs from p and from $\tilde{f}$ are used. If we remove one pair from $\tilde{f}$ for each of those evaluation paths to obtain $\hat{f}$ the partial permutation $w(p \cup \hat{f})$ will only have fixed points that $w(p)$ already had.

So we only have to find an upper bound for the number of evaluation paths using pairs from both p and $\tilde{f}$. This upper bound is attained if, for each occurrence of x in w and any pair of p, it gets completed to an evaluation path with all pairs from $\tilde{f}$. This gives us $|p| \cdot \#_x(w)$ as an upper bound, where $\#_x(w)$ is the number of occurrences of x and x^{-1} in w.

So in total at most $2l + l \cdot \#_x(w)$ pairs (a, b) of f' are such that $p \cup \{(a, b)\}$ is not a very good extension of p with respect to w.

This means that if we take S to consist of any $2k + k \cdot \#_x(w) + 1$ pairs of f' we have a set as desired. $\square$

We need and easily get the following stronger lemma.

Lemma III.35. *Let H be a cofinitary group, $f \in \mathrm{Sym}(\mathbb{N}) \setminus H$ such that $\langle H, f\rangle$ is a cofinitary group and $w_0, \ldots, w_n \in W_H$. Then for every $k \in \mathbb{N}$ there exists a finite set S of pairs from f such that for every injective map p with $|p|$ less than k there exists a pair $(a, b) \in S$ such that $p \cup \{(a, b)\}$ is a very good extension of p for all the words $w_0, \ldots, w_n$.*

Proof. By applying the method used in the first half of the proof of the last lemma $n + 1$ times we can find an infinite $f' \subseteq f$ such that none of $w_0(f'), \ldots, w_n(f')$ have fixed points. Then using the method in the second half of the proof of the last lemma also $n + 1$ times we can find how big a subset S of f' we have to choose. □

Now we are ready to state and prove the second theorem of this subsection.

Theorem III.36. $\Diamond({}^{\mathbb{N}}\mathbb{N}, =^{\infty})$ *implies* $\mathfrak{a}_g = \aleph_1$.

Proof. We use the same strategy as in the proof of the previous theorem: we define a function F whose $\Diamond({}^{\mathbb{N}}\mathbb{N}, =^{\infty})$-sequence helps us build a maximal cofinitary group $\langle\{g_\alpha : \alpha < \omega_1\}\rangle$.

By coding, as in the proof of Theorem III.32, we let its domain be the set of pairs $(\langle g_\alpha : \alpha < \delta\rangle, g)$ with $\delta < \omega_1$ and $\{g_\alpha : \alpha < \delta\} \cup \{g\}$ a family of permutations. This coding works on a club $C \subseteq \omega_1$, which is enough. Also by coding we let its range be ${}^{\mathbb{N}}(\mathbb{N} \cup {}^{<\omega}(\mathbb{N} \times \mathbb{N}))$. We also fix for every $\delta < \omega_1$ a bijection $e_\delta : \mathbb{N} \to \delta$.

For $\langle g_\alpha : \alpha < \delta\rangle$ a sequence of permutations we let $n \mapsto \tilde{w}_n$ be an enumeration of $W_{\langle\{g_\alpha : \alpha < \delta\}\rangle}$.

Now we can define F. On the levels $\delta < \omega_1$ where the chosen coding for the input does not work, define F to be any constant map. On the levels where the coding does work, define $F(\langle g_\alpha : \alpha < \delta\rangle, g)(n)$ to be either m, the least code for

$\big((k_0, g(k_0)), (k_1, g(k_1)), \ldots, (k_N, g(k_N))\big)$ such that for every injective partial map $p : \mathbb{N} \rightharpoonup \mathbb{N}$ with $|p| \leq 3n$ there is a pair $(k_i, g(k_i))$ coded in m such that $p \cup \{(k_i, g(k_i))\}$ is a very good extension of p with respect to all words $\tilde{w}_0, \ldots, \tilde{w}_n$, or 0 if such a code does not exist.

Note that by Lemma III.35, if $\{g_\alpha : \alpha < \delta\} \cup \{g\}$ generates a cofinitary group and $g \notin \langle \{g_\alpha : \alpha < \delta\} \rangle$, then there is such a code m. Also note that the function F is Borel (which can be shown in the same way we showed F in the proof of Theorem III.32 to be Borel).

Let $G : \omega_1 \to {}^{\mathbb{N}}\mathbb{N}$ be a $\diamondsuit({}^{\mathbb{N}}\mathbb{N}, =^\infty)$-sequence for this F. We define $G(\delta)(n)$ to be a *valid guess* for $\langle g_\alpha : \alpha < \delta \rangle$, a family of permutations that generates a cofinitary group, iff

- $G(\delta)(n) = \big((k_0, o_0), (k_1, o_1), \ldots, (k_N, o_N)\big)$ for some $k_i, o_i \in \mathbb{N}$ and $N \in \mathbb{N}$,
- all k_i are distinct,
- all o_i are distinct, and
- for every partial injective map $p : \mathbb{N} \rightharpoonup \mathbb{N}$ with $|p| \leq 3n$ there is a pair (k_i, o_i) such that $p \cup \{(k_i, o_i)\}$ is a very good extension of p with respect to all words $\tilde{w}_0, \ldots, \tilde{w}_n$.

Note that for any $\delta < \omega_1$, $n \in \mathbb{N}$, and any permutation g such that $g \notin \langle \{g_\alpha : \alpha < \delta\} \rangle$ and $\langle \{g_\alpha : \alpha < \delta\} \cup \{g\} \rangle$ is cofinitary, if $F(\langle g_\alpha : \alpha < \delta \rangle, g)(n) = G(\delta)(n)$ then $G(\delta)(n)$ is a valid guess for $\langle g_\alpha : \alpha < \delta \rangle$.

Now we use G to construct recursively $\langle g_\alpha : \alpha < \omega_1 \rangle$, a sequence of permutations which generates a maximal cofinitary group. So suppose $\langle g_\alpha : \alpha < \delta \rangle$ has been constructed. Then construct $g_\delta := \bigcup_{s \in \mathbb{N}} g_{\delta,s}$ recursively by:

P1. $g_{\delta,0} := \emptyset$,

P2. $g'_{\delta,s+1} := g_{\delta,s}$ if $G(\delta)(s)$ is not a valid guess for $\langle g_\alpha : \alpha < \delta \rangle$,

P3. $g'_{\delta,s+1} := g_{\delta,s} \cup \{(k_i, o_i)\}$ if $G(\delta)(s) = ((k_0, o_0), \ldots, (k_N, o_N))$ is a valid guess for $\langle g_\alpha : \alpha < \delta \rangle$ and i is least such that $p \cup \{(k_i, o_i)\}$ is a very good extension of p for all words $\tilde{w}_0, \ldots, \tilde{w}_n$,

P4. $g''_{\delta,s+1} := g'_{\delta,s+1} \cup \{(a, b)\}$ where a is the least number not in $\mathrm{dom}(g'_{\delta,s+1})$ and b is the least number such that $g'_{\delta,s+1} \cup \{(a, b)\}$ is a good extension of $g'_{\delta,s+1}$ for all words $\tilde{w}_0, \ldots, \tilde{w}_n$ (this b exists by the domain extension lemma), and

P5. $g_{\delta,s+1} := g''_{\delta,s+1} \cup \{(c, d)\}$ where d is the least number not in $\mathrm{ran}(g''_{\delta,s+1})$ and c is the least number such that $g''_{\delta,s+1} \cup \{(c, d)\}$ is a good extension of $g''_{\delta,s+1}$ with respect to all words $\tilde{w}_0, \ldots, \tilde{w}_n$ (this c exists by the range extension lemma).

Note that $|g_{\delta,s}|$ is at most $3s$ which means we can always perform step P3 when applicable.

Now g_δ is a permutation (by P4 and P5) such that $\{g_\alpha : \alpha < \delta\} \cup \{g_\delta\}$ generates a cofinitary group; this completes the construction of $\langle g_\alpha : \alpha < \omega_1 \rangle$.

It remains to see that this group is *maximal* cofinitary. Suppose, towards a contradiction, that there is $g \in \mathrm{Sym}(\mathbb{N})$ such that $g \notin \langle \{g_\alpha : \alpha < \omega_1\} \rangle$ and that $\langle \{g_\alpha : \alpha < \omega_1\}, g \rangle$ is a cofinitary group. Then the set

$$\{\delta < \omega_1 : F(\langle g_\alpha : \alpha < \delta \rangle, g) =^\infty G(\delta)\}$$

is stationary. Remember that we use a coding for the inputs of the function F, and note that the sequence $\delta \mapsto (\langle g_\alpha : \alpha < \delta \rangle, g)$ determines a path $f : \omega_1 \to 2$ in the tree ${}^{<\omega_1}2$. Now let $\delta \in c$ be a member of this set. Then $F(\langle g_\alpha : \alpha < \delta \rangle, g) =^\infty G(\delta)$, which means that for infinitely many n, the value $G(\delta)(n)$ is a valid guess for $\langle g_\alpha : \alpha < \delta \rangle$ and all pairs in $G(\delta)(n)$ belong to g. This means we hit g infinitely often with g_δ (by P5), which is a contradiction. $\square$

Combining Theorem III.32 and Theorem III.36 with

Theorem III.37 ([29])**.** $\diamondsuit({}^{\mathbb{N}}\mathbb{N}, =^{\infty})$ *is true in the Miller model.*

we have

Corollary III.38. $\mathfrak{a}_p = \mathfrak{a}_g = \aleph_1$ *is true in the Miller model.*

Then with $\mathfrak{g} \leq c(\mathrm{Sym}(\mathbb{N}))$, from [8] ($\mathfrak{g}$ is the groupwise density, for a definition see [5], and $c(\mathrm{Sym}(\mathbb{N}))$ is the cofinality of the symmetric group, see Definition I.35), and the fact, from [5], that the cardinal $\mathfrak{g}$ is $\aleph_2$ in the Miller model, we obtain the following theorem.

Theorem III.39. *In the Miller model* $\mathfrak{a}_p = \mathfrak{a}_g = \aleph_1 < \aleph_2 = c(\mathrm{Sym}(\mathbb{N}))$.

3.4.2 Template Forcing Result

In this subsection we prove the following theorem (for $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ see Definition I.35).

Theorem III.40. *The continuum hypothesis implies that for all regular cardinals* $\lambda > \mu > \aleph_1$ *with* $\lambda = \lambda^\omega$, *there is a forcing extension in which* $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mu$ *and* $\mathfrak{a}_g = 2^{\aleph_0} = \lambda$.

This proof is derived from the work of Jörg Brendle in [7] and suggestions from Tapani Hytinnen. We will show the details for the arguments Brendle only outlines, but refer to [7] for the arguments that are given completely there.

The Basic Forcing

Our basic forcing will be localization forcing. Here we collect the basic facts about this forcing for the convenience of the reader.

Definition III.41. For S a set, and $n \in \mathbb{N}$ define

- ${}^{<\omega}[S]$ to be the set of finite subsets of S.

- ${}^{\leq n}[S]$ to be the set of subsets of S of size $\leq n$.

- ${}^{n}[S]$ to be the set of size n subsets of S.

- ${}^{<\omega}(S)$ to be the set of finite sequences from S.

- ${}^{\mathbb{N}}S$ to be the set of functions from $\mathbb{N}$ to S.

The localization forcing notion $\mathbb{LOC} = \langle P, \leq \rangle$ is defined by

- P is the set of all pairs $(\sigma, \varphi) \in {}^{<\omega}({}^{<\omega}[\mathbb{N}]) \times {}^{\mathbb{N}}({}^{<\omega}[\mathbb{N}])$ such that $|\sigma(i)| \leq i$ for all $i < \mathrm{lh}(\sigma)$ and $|\varphi(i)| \leq \mathrm{lh}(\sigma)$ for all i.

- $(\tau, \psi) \leq (\sigma, \varphi)$ if $\mathrm{lh}(\tau) \geq \mathrm{lh}(\sigma)$, $\sigma \subseteq \tau$, $\varphi(j) \subseteq \tau(j)$ for all $\mathrm{lh}(\sigma) \leq j < \mathrm{lh}(\tau)$ and $\varphi(j) \subseteq \psi(j)$ for all j.

Note that here $\sigma \subseteq \tau$ means that $\sigma = \tau \restriction \mathrm{lh}(\sigma)$.

Lemma III.42. $\mathbb{LOC}$ *is σ-linked.*

Proof. A sufficient (but not necessary) condition for (σ_1, φ_1) and (σ_2, φ_2) to be compatible is:

$$\sigma_1 = \sigma_2 \ \wedge \ (\varphi_1(i) = \varphi_2(i) \text{ for all } i \leq 2\,\mathrm{lh}(\sigma_1)).$$

There are only countably many choices for σ and then only countably many for the part of φ that matters. □

A *slalom* is a function $\phi : \mathbb{N} \to {}^{<\omega}[\mathbb{N}]$ such that for all $n \in \mathbb{N}$ we have $|\phi(n)| \leq n$. We say a slalom *localizes a real* $f \in {}^{\mathbb{N}}\mathbb{N}$ if there is an $m \in \mathbb{N}$ such that for all $n \geq m$ we have $f(n) \in \phi(n)$.

Lemma III.43. $\mathbb{LOC}$ *adds a slalom which localizes all ground model reals.*

Proof. If G is $\mathbb{LOC}$ generic, then $\phi = \bigcup_{p \in G} \pi_0(p)$ is a function $\mathbb{N} \to {}^{<\omega}[\mathbb{N}]$ such that $|\phi(n)| \leq n$, i.e. a slalom.

For $f \in {}^{\mathbb{N}}\mathbb{N}$ define $D_f = \{p \in \mathbb{LOC} : p \Vdash f \text{ is localized by } \bigcup_{g \in \dot{G}} \pi_0(p)\}$ ($\dot{G}$ a name for the generic). Then D_f is dense: If $p = (\sigma, \varphi) \in \mathbb{LOC}$, then $(\sigma \hat{\ } \varphi(\mathrm{lh}(\sigma)), \varphi')$, with $\varphi'(n) = \varphi(n) \cup \{f(n)\}$, is an extension of p that is a member of D_f. □

Note that [2, page 106] defines localization forcing, **LOC**, to have underlying set $\{S \in {}^{\mathbb{N}}({}^{<\omega}[\mathbb{N}]) : \forall n |S(n)| \leq n \wedge \exists k, N \forall n \geq N |S(n)| \leq k\}$ with order $S \leq S'$ iff $\forall n [S'(n) \subseteq S(n)]$. We see that this forcing is equivalent to $\mathbb{LOC}$ by noting that the set $\mathcal{L}$ of those $S \in \mathbf{LOC}$ with for all $i < l$, $|S(i)| = i$, where l is the least number such that for all n we have $|S(n)| \leq l$, is dense in **LOC**. And the suborder $(\mathcal{L}, \leq)$ densely embeds in $\mathbb{LOC}$ by the embedding $F : \mathbf{LOC} \to \mathbb{LOC}$ defined by $F(S) = (\sigma, \varphi)$ with $\varphi = S$ and $\mathrm{lh}(\sigma) = l$, where l is the least number such that for all n we have $|S(n)| \leq l$ and $\sigma(i) = S(i)$ for $i \leq l$.

The reason for using this forcing here comes from the following characterizations of $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ given by Bartoszyński, see [2, Chapter 2].

Theorem III.44. $\mathrm{add}(\mathcal{N})$ *is the least cardinality of a family* $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ *such that there is no slalom localizing all members of* F.

$\mathrm{cof}(\mathcal{N})$ *is the least cardinality of a family* Φ *of slaloms such that every member of* ${}^{\mathbb{N}}\mathbb{N}$ *is localized by a member of* Φ.

The construction of the model for Theorem III.40 is by iterated forcing. The general structure of an iterated forcing construction consists of a system $\langle \mathbb{P}_i : i \in O \rangle$ of partial orders indexed by a directed set $(O, <)$ with maximal element m such that if $i < j$ then there is a complete embedding $p_{i,j} : \mathbb{P}_i \to \mathbb{P}_j$. Our forcing will be such that there is a cofinal set $C \subseteq O \setminus \{m\}$ of order type μ such that for $\alpha \in C$,

$\mathbb{P}_\alpha = \mathbb{Q}_\alpha * \mathbb{LOC}$ (the two step iteration of $\mathbb{Q}_\alpha$ and $\mathbb{LOC}$) for some $\mathbb{Q}_\alpha$ and for all $i < \alpha$ the embedding $p_{i,\alpha}$ maps into $\mathbb{Q}_\alpha$. The forcing will also have the property that for any $\mathbb{P}_m$ generic G and any real $r \in V[G]$, there is some $i < m$ such that $r \in V[\mathbb{P}_i \cap G]$.

This shows that the family of slaloms added at the coordinates in C gives a family Φ of slaloms in $V^{\mathbb{P}_m}$ such that any real is localized by a member of Φ, which with the theorem above shows that $\mathrm{cof}(\mathcal{N}) \leq \mu$ in $V^{\mathbb{P}_m}$.

Using similar ideas we see that $\mathrm{add}(\mathcal{N}) \geq \mu$ in $V^{\mathbb{P}_m}$. If F is a family of reals of size $< \mu$ in $V[G]$, then it already appears in $V[G \cap \mathbb{P}_i]$ for some $i < m$. But then it is localized by the slalom that gets added by any forcing $\mathbb{P}_\alpha$ with $\alpha > i$.

This shows that after we have constructed a forcing that has the properties claimed here, in the generic extension all cardinal numbers in Cichoń's diagram will be equal to μ (all forcings used are c.c.c. so that cardinals in V and $V[G]$ are the same).

We will then show by an isomorphism-of-names argument that in the forcing extension no cofinitary group of cardinality less than $2^{\aleph_0}$ (which will equal λ by a counting-of-names argument) and bigger or equal to μ can exist. This shows that $\mathfrak{a}_g = 2^{\aleph_0}$ using the following result by Brendle, Spinas, and Zhang (for $\mathrm{non}(\mathcal{M})$ see Definition I.35).

Theorem III.45 ([9])**.** $\mathfrak{a}_g \geq \mathrm{non}(\mathcal{M})$.

This Forcing Along a Template

Here we introduce templates and describe how to iterate our basic forcing along them. The notion of template forcing was introduced by Saharon Shelah in [31]. Jörg Brendle in [7] wrote an introduction to this theory. In [7] Brendle indicates how to change his definitions and proofs to work for localization forcing; we fill in the details.

In the following L will always denote a linear order $(L, \leq)$, and L_x the initial segment of L determined by x, i.e. $L_x = \{y \in L : y < x\}$.

The directed system of posets will be indexed by $(\mathcal{P}(L), \subseteq)$ for some L. So for each subset A of L we need to construct a poset. This is done by using a system of subsets of A along which we can recursively define the poset, and which has enough structure to prove the properties we need. In this section we will show how to do this. The order L (and template $\mathcal{I}$) we use will be defined in the next section.

Definition III.46. A *template* is a pair $(L, \mathcal{I})$, with $\mathcal{I} \subseteq \mathcal{P}(L)$ satisfying:

1. $\emptyset, L \in \mathcal{I}$;
2. $\mathcal{I}$ is closed under finite unions and finite intersections;
3. if $y, x \in L$ are such that $y < x$, then there is an $A \in \mathcal{I}$ such that $A \subseteq L_x$ and $y \in A$;
4. for all $A \in \mathcal{I}$ and $x \in L \setminus A$ the set $A \cap L_x$ belongs to $\mathcal{I}$;
5. $\mathcal{I}$ is wellfounded with respect to $\subseteq$.

Since $\mathcal{I}$ is wellfounded we can define a rank function $\mathrm{Dp} : \mathcal{I} \to \mathsf{ON}$ by $\mathrm{Dp}(\emptyset) := 0$ and $\mathrm{Dp}(A) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathcal{I} \wedge B \subsetneq A\}$.

We want to use one template to generate all the posets in the directed system. This is done by defining, in the next section, one big template and, for the smaller indices, using induced templates as defined next.

If $L' \subseteq L$ we define the *induced template* $(L', \mathcal{I} \restriction L')$ by $\mathcal{I} \restriction L' := \{A \cap L' : A \in \mathcal{I}\}$. An induced template is in fact a template: properties 1—4 are easy to verify. To verify 5 suppose, towards a contradiction, that A'_n, $n \in \mathbb{N}$, is an infinite strictly decreasing sequence in $\mathcal{I} \restriction L'$, then for each A'_n there is an $A_n \in \mathcal{I}$ such that $A_n \cap L' = A'_n$. Then $\tilde{A}_n = \bigcap_{k \leq n} A_n$ will be an infinite strictly decreasing sequence in $\mathcal{I}$.

For p a finite sequence with domain contained in the order L, we write $\mathrm{md}(p)$ for $\max \mathrm{dom}(p)$.

Definition III.47. The poset $\mathbb{P}(L, \mathcal{I})$ has underlying set recursively defined as follows:

1. $\mathbb{P}(\emptyset, \mathcal{I} \restriction \emptyset) = \{\emptyset\}$;

2. $p \in \mathbb{P}(L, \mathcal{I})$ if p is a finite sequence with $\mathrm{dom}(p) \subseteq L$ and there exists $B \in \mathcal{I}$ such that $B \subseteq L_{\mathrm{md}(p)}$, $p \restriction L_{\mathrm{md}(p)} \in \mathbb{P}(B, \mathcal{I} \restriction B)$ and $p(\mathrm{md}(p)) = (\sigma, \dot{\varphi})$, with $\sigma \in {}^{<\omega}({}^{<\omega}[\mathbb{N}])$ and $\dot{\varphi}$ a $\mathbb{P}(B, \mathcal{I} \restriction B)$ name such that $p \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B, \mathcal{I} \restriction B)} (\check{\sigma}, \dot{\varphi}) \in \mathbb{LOC}$.

Note that in this definition in particular we have that $p \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B, \mathcal{I} \restriction B)} \dot{\varphi} \in {}^{\mathbb{N}}([\mathbb{N}]^{\leq \mathrm{lh}(\check{\sigma})})$. Also we don't use all names for forcing conditions in $\mathbb{LOC}$, since the first projection of any $p(x)$ is an element of ${}^{<\omega}({}^{<\omega}[\mathbb{N}])$, not a name for such an element; the set of names we use is however dense in $\mathbb{LOC}$.

Definition III.48. For $p, q \in \mathbb{P}(L, \mathcal{I})$, we define $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and

1. if $\mathrm{md}(p) = \mathrm{md}(q)$, then there is a $B \in \mathcal{I}$ with $B \subseteq L_{\mathrm{md}(p)}$ such that $\models q \restriction L_{\mathrm{md}(p)} \leq_{\mathbb{P}(B, \mathcal{I} \restriction B)} p \restriction L_{\mathrm{md}(p)}$ and $q \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B, \mathcal{I} \restriction B)} q(\mathrm{md}(p)) \leq_{\mathbb{LOC}} p(\mathrm{md}(p))$;

2. if $\mathrm{md}(q) > \mathrm{md}(p)$ then there is a $B \in \mathcal{I}$ with $\mathrm{dom}(q \restriction L_{\mathrm{md}(q)}) \subseteq B \subseteq L_{\mathrm{md}(q)}$ such that $q \restriction L_{\mathrm{md}(q)} \leq_{\mathbb{P}(B, \mathcal{I} \restriction B)} p$.

Note that in the first item $q \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B, \mathcal{I} \restriction B)} q(\mathrm{md}(p)) \leq_{\mathbb{LOC}} p(\mathrm{md}(p))$ is shorthand for the corresponding statement with checks introduced on the first coordinates of $q(\mathrm{md}(p))$ and $p(\mathrm{md}(p))$.

Of course at every recursive step we can take the usual action (finding a V_β containing names for all possible $\varphi : \mathbb{N} \to {}^{<\omega}[\mathbb{N}]$ in the generic extension) to ensure that we in fact end up with a set of forcing conditions.

Now since in defining the order there was an existential quantifier over $B \subseteq A \cap L_x$ it is not immediately clear that this defines an order at all. Reflexivity is clear (antisymmetry is false, but as is usual we can look at the poset modulo the relation $p \leq q$ and $q \leq p$), but transitivity is not clear. It is conceivable that $r \leq q$ and $q \leq p$ use different witnesses that can not be unified to give $r \leq p$. The following lemma shows that this does not happen.

Lemma III.49. $\mathbb{P}(L, \mathcal{I})$ *is a poset.*

The next lemma shows that this sort of forcing can be used for an iterated forcing argument.

Lemma III.50 (Completeness of Embeddings)**.** *For any* $A \subseteq B \subseteq L$ *we have that* $\mathbb{P}(A, \mathcal{I} \restriction A)$ *completely embeds into* $\mathbb{P}(B, \mathcal{I} \restriction B)$*, written* $\mathbb{P}(A, \mathcal{I} \restriction A) <\!\circ\, \mathbb{P}(B, \mathcal{I} \restriction B)$*.*

We will in fact prove both these lemmas at the same time by recursion, showing that an apparently stronger form of complete embedding is true (we currently don't know whether this notion really is stronger).

Lemma III.51 ([7, Lemma 4.4]; similar to [7, Lemma 1.1])**.** *For any* $B \in \mathcal{I}$ *and* $A \subseteq B$*,* $\mathbb{P}(B, \mathcal{I} \restriction B)$ *is a partial order, and for any* $p \in \mathbb{P}(B, \mathcal{I} \restriction B)$ *there exists a* $\mathrm{red}(p, A, B) \in \mathbb{P}(A, \mathcal{I} \restriction A)$ *such that for any* $D \in \mathcal{I}$ *and* $C \subseteq D$ *such that* $B \subseteq D$ *and* $B \cap C = A$ *if* $r \leq_{\mathbb{P}(C, \mathcal{I} \restriction C)} \mathrm{red}(p, A, B)$ *then* r *and* p *are compatible in* $\mathbb{P}(D, \mathcal{I} \restriction D)$*.*

Proof. By induction on α we will show that

1. for all templates $(L, \mathcal{I})$ with $\mathrm{Dp}(L) \leq \alpha$, the poset $\mathbb{P}(L, \mathcal{I})$ is transitive,

2. for all templates $(L, \mathcal{I})$ with $\mathrm{Dp}(L) \leq \alpha$ and $A \subseteq L$,

$$\mathbb{P}(A, \mathcal{I} \restriction A) \subseteq \mathbb{P}(L, \mathcal{I}),$$

3. for all templates $(L, \mathcal{I})$ with $\mathrm{Dp}(L) \leq \alpha$, all $B \in \mathcal{I}$, all $A \subseteq B$, and all $p \in \mathbb{P}(B, \mathcal{I} \restriction B)$ there exists an element $\mathrm{red}(p, \mathcal{I} \restriction B, A) \in \mathbb{P}(A, \mathcal{I} \restriction A)$ such that $\mathrm{dom}(\mathrm{red}(p, \mathcal{I} \restriction B, A)) = \mathrm{dom}(p) \cap A$ and for all $x \in \mathrm{dom}(p) \cap A$ we have $\pi_0(\mathrm{red}(p, \mathcal{I} \restriction B, A)(x)) = \pi_0(p(x))$ (note: $\mathrm{red}(p, \mathcal{I} \restriction B, A)$ depends only on p, $\mathcal{I} \restriction B$ and A, *not* on the part of the template outside of $\mathcal{I} \restriction B$),

4. for all templates $(L, \mathcal{I})$ with $\mathrm{Dp}(L) \leq \alpha$ and all $D \in \mathcal{I}$, $B, C \subseteq D$, and $p \in \mathbb{P}(B, \mathcal{I} \restriction B)$, that (write A for $B \cap C$) for all $q \in \mathbb{P}(C, \mathcal{I} \restriction C)$,

$$q \leq_{\mathbb{P}(C, \mathcal{I} \restriction C)} \mathrm{red}(p, \mathcal{I} \restriction B, A) \quad \Rightarrow \quad p \parallel_{\mathbb{P}(D, \mathcal{I} \restriction D)} q.$$

We first show 1. Let $p, q, r \in \mathbb{P}(L, \mathcal{I})$ such that $r \leq q$ and $q \leq p$. We then need to analyze cases depending on the orders of $\mathrm{md}(r), \mathrm{md}(q)$, and $\mathrm{md}(p)$. We'll show one; the others are analogous. The main idea here is that a template is closed under finite unions so that we can unify the witness to the order.

So assume $\mathrm{md}(r) = \mathrm{md}(q) = \mathrm{md}(p)$. Then there is a $B^{p,q} \in \mathcal{I}$ with $B^{p,q} \subseteq L_{\mathrm{md}(p)}$ such that $q \restriction L_{\mathrm{md}(p)} \leq_{\mathbb{P}(B^{p,q}, \mathcal{I} \restriction B^{p,q})} p \restriction L_{\mathrm{md}(p)}$ and $q \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B^{p,q}, \mathcal{I} \restriction B^{p,q})} q(\mathrm{md}(p)) \leq_{\mathbb{LOC}} p(\mathrm{md}(p))$. There is a similar set $B^{q,r}$ obtained from the order $r \leq q$. Then set $B^{p,r} = B^{p,q} \cup B^{q,r}$, and note that by induction hypothesis (since $\mathrm{md}(p) \notin B^{p,r}$, $\mathrm{Dp}(B^{p,r}) < \mathrm{Dp}(L)$) we have that $\mathbb{P}(B^{p,q}, \mathcal{I} \restriction B^{p,q}), \mathbb{P}(B^{q,r}, \mathcal{I} \restriction B^{q,r}) <\circ \mathbb{P}(B^{p,r}, \mathcal{I} \restriction B^{p,r})$. From this we get that $r \restriction L_{\mathrm{md}(p)} \leq_{\mathbb{P}(B^{p,r}, \mathcal{I} \restriction B^{p,r})} p \restriction L_{\mathrm{md}(p)}$ and $r \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B^{p,r}, \mathcal{I} \restriction B^{p,r})} r(\mathrm{md}(p)) \leq_{\mathbb{LOC}} p(\mathrm{md}(p))$, which witnesses that $r \leq p$ as was to be shown.

Now we show 2. Let $A \subset L$ and $p \in \mathbb{P}(A, \mathcal{I} \restriction A)$. We need to show that

$p \in \mathbb{P}(L, \mathcal{I})$. The main idea is to unfold one step of the construction that shows that $p \in \mathbb{P}(A, \mathcal{I} \upharpoonright A)$, and then use a similar construction step to build $p \in \mathbb{P}(L, \mathcal{I})$.

$p \in \mathbb{P}(A, \mathcal{I} \upharpoonright A)$ means there is $B \in \mathcal{I} \upharpoonright A$ such that $B \subseteq L_{\mathrm{md}(p)}$, $p \upharpoonright L_{\mathrm{md}(p)} \in \mathbb{P}(B, \mathcal{I} \upharpoonright B)$, and $p \upharpoonright L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B, \mathcal{I} \upharpoonright B)} p(\mathrm{md}(p)) \in \mathbb{LOC}$.

$B \in \mathcal{I} \upharpoonright A$ means that $B = B' \cap A$ for some $B' \in \mathcal{I}$. Since $\mathrm{md}(p) \in A \setminus B$ we have $\mathrm{md}(p) \notin B'$ and we can (using the definition of template) assume that $B' \subseteq L_{\mathrm{md}(p)}$. By the induction hypothesis we have $\mathbb{P}(B, \mathcal{I} \upharpoonright B) <\circ \mathbb{P}(B', \mathcal{I} \upharpoonright B')$, which gives us that $p \upharpoonright L_{\mathrm{md}(p)} \in \mathbb{P}(B', \mathcal{I} \upharpoonright B')$ and $p \upharpoonright L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B', \mathcal{I} \upharpoonright B')} p(\mathrm{md}(p)) \in \mathbb{LOC}$. This in turn shows $p \in \mathbb{P}(L, \mathcal{I})$.

Now we show 3. Assume that $\mathrm{red}(p, I \upharpoonright B, A)$ has already been defined for B with $\mathrm{Dp}(B) < \alpha$. So all that remains at this stage is to define it for $B = L$. The main idea is to use the reduction already defined for $p \upharpoonright L_{\mathrm{md}(p)}$, and for the reduction at $\mathrm{md}(p)$ (if it needs to be defined) use a reduction of $\pi_1(\mathrm{md}(p))$.

Let $A \subseteq L$ and $p \in \mathbb{P}(L, \mathcal{I})$. There exists $B^{\mathrm{md}(p)} \in \mathcal{I}$ such that $B^{\mathrm{md}(p)} \subseteq L_{\mathrm{md}(p)}$, $p \upharpoonright L_{\mathrm{md}(p)} \in \mathbb{P}(B^{\mathrm{md}(p)}, \mathcal{I} \upharpoonright B^{\mathrm{md}(p)})$, and $p \upharpoonright L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B^{\mathrm{md}(p)}, \mathcal{I} \upharpoonright B^{\mathrm{md}(p)})} p(\mathrm{md}(p)) \in \mathbb{LOC}$. Set $A^{\mathrm{md}(p)} = A \cap B^{\mathrm{md}(p)}$.

If $\mathrm{md}(p) \notin A$, define $\mathrm{red}(p, \mathcal{I}, A) := \mathrm{red}(p \upharpoonright L_{\mathrm{md}(p)}, \mathcal{I} \upharpoonright B^{\mathrm{md}(p)}, A^{\mathrm{md}(p)})$. Since $\mathrm{Dp}(B^{\mathrm{md}(p)}) < \alpha$ we have by induction hypothesis that $\mathbb{P}(A^{\mathrm{md}(p)}, \mathcal{I} \upharpoonright A^{\mathrm{md}(p)}) <\circ \mathbb{P}(A, \mathcal{I} \upharpoonright A)$, so that indeed $\mathrm{red}(p, \mathcal{I}, A) \in \mathbb{P}(A, \mathcal{I} \upharpoonright A)$.

Otherwise, define $\mathrm{red}(p, \mathcal{I}, A) \upharpoonright L_{\mathrm{md}(p)} := \mathrm{red}(p \upharpoonright L_{\mathrm{md}(p)}, \mathcal{I} \upharpoonright B^{\mathrm{md}(p)}, A^{\mathrm{md}(p)})$, and $\mathrm{red}(p, \mathcal{I}, A)(\mathrm{md}(p)) := (\sigma, \dot{\varphi})$, where $p(\mathrm{md}(p)) = (\sigma, \dot{\psi})$ and $\dot{\varphi}$ is the reduction of the $\mathbb{P}(B^{\mathrm{md}(p)}, \mathcal{I} \upharpoonright B^{\mathrm{md}(p)})$-name $\dot{\psi}$ to $\mathbb{P}(A^{\mathrm{md}(p)}, \mathcal{I} \upharpoonright A^{\mathrm{md}(p)})$ defined as follows:

We know that $\mathbb{P}(A^{\mathrm{md}(p)}, \mathcal{I} \upharpoonright A^{\mathrm{md}(p)}) <\circ \mathbb{P}(B^{\mathrm{md}(p)}, \mathcal{I} \upharpoonright B^{\mathrm{md}(p)})$. Consider the complete Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ which are the regular open algebras of these respective posets.

Note that the $\mathbb{B}$-name $\dot{\psi}$ is completely determined by the Boolean values $[\![\check{\sigma} \subseteq \dot{\psi}]\!]$ where $\sigma \in {}^{<\omega}({}^{<\omega}[\mathbb{N}])$. $p \restriction L_{\mathrm{md}(p)} \Vdash \dot{\psi} \in {}^{\mathbb{N}}({}^{\mathrm{lh}(\check{\sigma})}[\mathbb{N}])$ is equivalent to

$$\forall \sigma' \in {}^{<\omega}({}^{<\omega}[\mathbb{N}]) \left(\exists i \in \mathrm{dom}(\sigma')\ |\sigma'(i)| > \mathrm{lh}(\sigma)\right) \to p \restriction L_{\mathrm{md}(p)} \wedge [\![\check{\sigma}' \subseteq \dot{\psi}]\!] = 0.$$

For $\sigma \in {}^{<\omega}({}^{<\omega}[\mathbb{N}])$, define a_σ to be the projection of $[\![\check{\sigma} \subseteq \dot{\psi}]\!]$ to $\mathbb{A}$ (this projection is $\bigwedge\{a \in \mathbb{A} : [\![\check{\sigma} \subseteq \dot{\psi}]\!] \leq a\}$). Enumerate ${}^{<\omega}[\mathbb{N}]$ as $\langle c_k : k \in \mathbb{N}\rangle$, and recursively on the length of σ define $[\![\check{\sigma} \subseteq \dot{\varphi}]\!]$ to be

$$\mathrm{red}(p, \mathcal{I} \restriction B, A) \restriction L_{\mathrm{md}(p)} \wedge [\![(\sigma \restriction (\mathrm{lh}(\sigma) - 1))\check{} \subseteq \dot{\varphi}]\!] \wedge (a_\sigma \setminus \bigvee_{k<l} a_{(\sigma \restriction \mathrm{lh}(\sigma)-1)^\frown c_k}),$$

where l is such that $\sigma(\mathrm{lh}(\sigma)) = c_l$.

Note that $[\![\check{\sigma}_1 \subseteq \dot{\varphi}]\!] \wedge [\![\check{\sigma}_2 \subseteq \dot{\varphi}]\!] = 0$ if there is an i such that $\sigma_1(i) \neq \sigma_2(i)$ or if there is an i such that $|\sigma_1(i)| > \mathrm{lh}(\sigma)$, since this is true for $\dot{\psi}$. Also $\langle [\![(\sigma^\frown c_k)\check{} \subseteq \dot{\varphi}]\!] : k \in \mathbb{N}\rangle$ is a maximal antichain below $[\![\check{\sigma} \subseteq \dot{\varphi}]\!]$.

Now we show that the reduction works as advertised, i.e. show 4. This is just a matter of using the induction hypothesis and the properties of templates which allow us to construct the right sets to consider conditions in and over.

So let $D \in \mathcal{I}$, $B, C \subseteq D$, and $p \in \mathbb{P}(B, \mathcal{I} \restriction B)$. Write A for $B \cap C$. Let $B^{\mathrm{md}(p)}$ and $A^{\mathrm{md}(p)}$ be as in the definition of $\mathrm{red}(p, \mathcal{I} \restriction B, A)$ ($A^{\mathrm{md}(p)} = A$ if $\mathrm{md}(p) \notin A$). Let $q \in \mathbb{P}(C, \mathcal{I} \restriction C)$ be such that $q \leq_{\mathbb{P}(C,\mathcal{I}\restriction C)} \mathrm{red}(p, \mathcal{I} \restriction B, A)$. We have to show that q and p are compatible in $\mathbb{P}(D, \mathcal{I} \restriction D)$.

If[1] $\mathrm{md}(p) \notin A$, then also $\mathrm{md}(p) \notin C$. We will use the induction hypothesis to find an r (with suitable domain) that witnesses that $q \restriction L_{\mathrm{md}(p)}$ and $\mathrm{red}(p, \mathcal{I} \restriction B, A) \restriction L_{\mathrm{md}(p)}$ are compatible. Then we show that the element $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\} \cup q \restriction \{y \in L : \mathrm{md}(p) < y\}$ shows that p and q are compatible.

Using the definition for $q \in \mathbb{P}(C, \mathcal{I} \restriction C)$, we can find a set $C^q \in \mathcal{I}$ with $C^q \subseteq L_{\min\{x \in L : x \in \mathrm{dom}(q) \wedge x > \mathrm{md}(p)\}}$ (if there is an $x \in \mathrm{dom}(q)$ with $x > \mathrm{md}(p)$, otherwise let

[1] We have underlined this and the next case to make them easier to find.

$C^q = C$) such that $q \restriction L_{\mathrm{md}(p)} \in \mathbb{P}(C^q, \mathcal{I} \restriction C^q)$. Since C^q might be somewhat too big for our purposes, we unfold the definition of q one step further and then refold it to a smaller set.

Since $q \restriction L_{\mathrm{md}(p)} \in \mathbb{P}(C^q, \mathcal{I} \restriction C^q)$, using the definition there exists $C^{q\prime} \in \mathcal{I} \restriction C^q$ such that $C^{q\prime} \subseteq L_{\mathrm{md}(q \restriction L_{\mathrm{md}(p)})}$, $q \restriction L_{\mathrm{md}(q \restriction L_{\mathrm{md}(p)})} \in \mathbb{P}(C^{q\prime}, \mathcal{I} \restriction C^{q\prime})$ and $q \restriction L_{\mathrm{md}(q \restriction L_{\mathrm{md}(p)})} \Vdash_{\mathbb{P}(C^{q\prime}, \mathcal{I} \restriction C^{q\prime})} q(\mathrm{md}(q \restriction L_{\mathrm{md}(p)})) \in \mathbb{LOC}$. Now there exists $C^{\mathrm{md}(p)} \subseteq L_{\mathrm{md}(p)}$ with $C^{q\prime} \subseteq C^{\mathrm{md}(p)}$, $A^{\mathrm{md}(p)} \subseteq C^{\mathrm{md}(p)}$, and $\mathrm{md}(q \restriction L_{\mathrm{md}(p)}) \in C^{\mathrm{md}(p)}$. Then $q \in \mathbb{P}(C^{\mathrm{md}(p)}, \mathcal{I} \restriction C^{\mathrm{md}(p)})$.

Setting $D^{\mathrm{md}(p)} = C^{\mathrm{md}(p)} \cup B^{\mathrm{md}(p)}$, we can find by induction hypothesis an $r \in \mathbb{P}(D^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)})$ such that $r \leq_{\mathbb{P}(D^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)})} q \restriction L_{\mathrm{md}(p)}, p \restriction L_{\mathrm{md}(p)}$.

Since $B^{\mathrm{md}(p)} \subseteq D^{\mathrm{md}(p)}$, we have $\mathbb{P}(B^{\mathrm{md}(p)}, \mathcal{I} \restriction B^{\mathrm{md}(p)}) <\circ \mathbb{P}(D^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)})$. Then since $r \leq p \restriction L_{\mathrm{md}(p)}$, we have that $r \Vdash_{\mathbb{P}(D^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)})} p(\mathrm{md}(p)) \in \mathbb{LOC}$.

Now if $\{y \in L : y \in \mathrm{dom}(q) \wedge \mathrm{md}(p) < y\}$ is empty, then from the above we find, by the definitions, that $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\} \in \mathbb{P}(D, \mathcal{I} \restriction D)$ and $r \leq p, q$. Otherwise we first have to enlarge $D^{\mathrm{md}(p)}$ to a set containing C^q so that we can from there on use the way q was constructed to construct $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\} \cup q \restriction \{y \in L : y \in \mathrm{dom}(q) \wedge \mathrm{md}(p) < y\}$.

Since $\mathrm{md}(p) < \min\{y \in L : y \in \mathrm{dom}(q) \wedge \mathrm{md}(p) < y\}$, there exists $E \in \mathcal{I}$ with $E \subseteq L_{\{y \in L : y \in \mathrm{dom}(q) \wedge \mathrm{md}(p) < y\}}$ and $\mathrm{md}(p) \in E$. Set $E' = E \cup C^q \cup D^{\mathrm{md}(p)}$. Then with the above we get $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\} \in \mathbb{P}(E', \mathcal{I} \restriction E')$. By the induction hypothesis $\mathbb{P}(C^q, \mathcal{I} \restriction C^q) <\circ \mathbb{P}(E', \mathcal{I} \restriction E')$, and using the definition of order we have $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\} \leq_{\mathbb{P}(E', \mathcal{I} \restriction E')} q \restriction q \restriction L_{\mathrm{md}(p)}$ and $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\} \leq_{\mathbb{P}(E', \mathcal{I} \restriction E')} p$.

We can now add the next element from the domain of q to $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\}$. After iteratively adding the elements from the domain of q, we find $r \cup \{(\mathrm{md}(p),$

$p(\mathrm{md}(p)))\} \cup q \restriction \{y \in L : \mathrm{md}(p) < x\} \leq_{\mathbb{P}(\bar{C},\mathcal{I}\restriction\bar{C})} q, p$ for a $\bar{C}$ containing C. If when adding $\mathrm{md}(q)$, we take the domain set to be all of D, then we get $r \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\} \cup q \restriction \{y \in L : \mathrm{md}(p) < x\} \leq_{\mathbb{P}(D,\mathcal{I}\restriction D)} q, p$ as desired.

If $\mathrm{md}(p) \in A$, then $q \leq_{\mathbb{P}(C,\mathcal{I}\restriction C)} \mathrm{red}(p, \mathcal{I} \restriction B, A)$ is witnessed by $C^{\mathrm{md}(p)} \in \mathcal{I}$ such that $A^{\mathrm{md}(p)} \subseteq C^{\mathrm{md}(p)} \subseteq L_{\mathrm{md}(p)}$, $q \restriction L_{\mathrm{md}(p)} \leq_{\mathbb{P}(C^{\mathrm{md}(p)},\mathcal{I}\restriction C^{\mathrm{md}(p)})} \mathrm{red}(p, \mathcal{I} \restriction B, A) \restriction L_{\mathrm{md}(p)}$, and $q \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(C^{\mathrm{md}(p)},\mathcal{I}\restriction C^{\mathrm{md}(p)})} q(\mathrm{md}(p)) \leq_{\mathbb{LOC}} \mathrm{red}(p, \mathcal{I} \restriction B, A)(\mathrm{md}(p))$ and from there by possibly extending q.

Since $\mathrm{red}(p, \mathcal{I} \restriction B, a) \restriction L_{\mathrm{md}(p)} = \mathrm{red}(p \restriction L_{\mathrm{md}(p)}, \mathcal{I} \restriction B^{\mathrm{md}(p)}, A^{\mathrm{md}(p)})$ and $C^{\mathrm{md}(p)} \cap B^{\mathrm{md}(p)} = A^{\mathrm{md}(p)}$, we can apply the induction hypothesis with $D^{\mathrm{md}(p)} = B^{\mathrm{md}(p)} \cup C^{\mathrm{md}(p)}$ to obtain $r^{\mathrm{md}(p)} \in \mathbb{P}(D^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)})$ such that $r^{\mathrm{md}(p)} \leq_{\mathbb{P}(D^{\mathrm{md}(p)},\mathcal{I}\restriction D^{\mathrm{md}(p)})} q \restriction L_{\mathrm{md}(p)}, p \restriction L_{\mathrm{md}(p)}$.

Write $q(\mathrm{md}(p)) = (\sigma_q, \dot{\varphi}_q)$ and remember that $\mathrm{red}(p, \mathcal{I} \restriction B, A)(\mathrm{md}(p)) = (\sigma_p, \dot{\varphi})$. Then

$$(*) \qquad r^{\mathrm{md}(p)} \Vdash_{\mathbb{P}(D^{\mathrm{md}(p)},\mathcal{I}\restriction D^{\mathrm{md}(p)})} (\check{\sigma}_q, \dot{\varphi}_q) \leq_{\mathbb{LOC}} (\check{\sigma}_p, \dot{\varphi}).$$

Therefore $\sigma_p \subseteq \sigma_q$ and $r^{\mathrm{md}(p)} \Vdash (\forall i \dot{\varphi}(i) \subseteq \dot{\varphi}_q(i)) \wedge \forall i(\mathrm{lh}(\check{\sigma}_p) < i \leq \mathrm{lh}(\check{\sigma}_q) \rightarrow \dot{\varphi}(i) \subseteq \check{\sigma}_q(i))$.

It is enough to find a condition $d^{\mathrm{md}(p)}$ in $\mathbb{P}(D^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)})$ such that $d^{\mathrm{md}(p)} \leq r^{\mathrm{md}(p)}$ and $s^{\mathrm{md}(p)} \Vdash (\check{\sigma}_q, \dot{\varphi}_q) \leq_{\mathbb{LOC}} (\sigma_p, \dot{\psi})$. Then by extending $d^{\mathrm{md}(p)}$ first by $(\mathrm{md}(p), (\sigma_q, \dot{\varphi}_q))$ and then in the same way q is extended we find the condition that shows p and q are compatible.

Write $\Sigma = \{\sigma \in {}^{<\omega}({}^{<\omega}[\mathbb{N}]) : \forall i(\mathrm{lh}(\sigma) \leq i < \mathrm{lh}(\sigma_q) \rightarrow \sigma(i) \subseteq \sigma_q(i))\}$, then by (*)

$$r^{\mathrm{md}(p)} \leq \bigvee_{\sigma \in \Sigma} [\![\sigma \subseteq \dot{\varphi}]\!] .$$

Therefore

$$\mathrm{red}(r^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)}, A^{\mathrm{md}(p)}) \leq \bigvee_{\sigma \in \Sigma} [\![\sigma \subseteq \dot{\varphi}]\!] .$$

The right hand side of this inequality is a reduction of $\bigvee_{\sigma\in\Sigma} [\![\sigma \subseteq \dot{\psi}]\!]$, and therefore there is a $b^{\mathrm{md}(p)} \in \mathbb{P}(B^{\mathrm{md}(p)}, \mathcal{I} \restriction B^{\mathrm{md}(p)})$ such that

$$b^{\mathrm{md}(p)} \leq_{\mathbb{P}(B^{\mathrm{md}(p)},\mathcal{I}\restriction B^{\mathrm{md}(p)})} r^{\mathrm{md}(p)}, \bigvee_{\sigma\in\Sigma} [\![\sigma \subseteq \dot{\psi}]\!]$$

By induction hypothesis this gives a condition $d^{\mathrm{md}(p)} \in \mathbb{P}(D^{\mathrm{md}(p)}, \mathcal{I} \restriction D^{\mathrm{md}(p)})$ such that $d^{\mathrm{md}(p)} \leq^{\mathbb{P}(D^{\mathrm{md}(p)},\mathcal{I}\restriction D^{\mathrm{md}(p)})} \bigvee_{\sigma\in\Sigma} [\![\sigma \subseteq \dot{\psi}]\!] , r^{\mathrm{md}(p)}$.

Therefore $d^{\mathrm{md}(p)} \Vdash \forall i(\mathrm{lh}(\check{\sigma_p}) \leq i < \mathrm{lh}(\check{\sigma_q}) \rightarrow \dot{\psi}(i) \subseteq \sigma_q(i))$ and $d^{\mathrm{md}(p)} \Vdash \forall i \dot{\psi}(i) \subseteq \dot{\varphi}_q(i)$; that is $d^{\mathrm{md}(p)} \Vdash (\check{\sigma_q}, \dot{\varphi}) \leq_{\mathbb{LOC}} (\check{\sigma_p}, \dot{\psi})$. Which is as was to be shown. □

We now show that the properties of this poset that establish that the forcing preserves cardinals and that any real in the forcing extension is already added by a small forcing completely embedded in it.

Definition III.52. A poset $\mathbb{P}$ satisfies *Knaster's condition* if every uncountable subset of it has an uncountable subset of pairwise compatible elements.

A poset satisfying Knaster's condition is clearly c.c.c.

Lemma III.53 ([7, Lemma 4.5])**.** *For all templates $(L, \mathcal{I})$, $\mathbb{P}(L, \mathcal{I})$ satisfies Knaster's condition.*

Proof. Let $\mathcal{A} \subseteq \mathbb{P}(L, \mathcal{I})$ be an uncountable set.

We first show by induction on the rank of L that the set of $p \in \mathbb{P}(L, \mathcal{I})$ such that for all $x \in \mathrm{dom}(p)$ there are a $B \subseteq L_x$ and $\tau \in {}^{2\,\mathrm{lh}(\pi_0(p(x)))}({}^{\leq \mathrm{lh}(\pi_0(p(x)))}[\mathbb{N}])$ such that $p \restriction L_x \Vdash_{\mathbb{P}(B,\mathcal{I}\restriction B)} \tau \subseteq \pi_1(p(x))$ is dense. These are the conditions where all the second projections at points in the domain are determined up to twice the length of the first coordinate.

For $p \in \mathbb{P}(L, \mathcal{I})$ there is a $B \subseteq L_{\mathrm{md}(p)}$ such that $p \restriction L_{\mathrm{md}(p)} \Vdash_{\mathbb{P}(B,\mathcal{I}\restriction B)} \pi_1(p(x)) \in {}^{\mathbb{N}}({}^{\leq \mathrm{lh}(\pi_0(p(\mathrm{md}(p))))}[\mathbb{N}])$. So there is a $q \in \mathbb{P}(B, \mathcal{I} \restriction B)$ and $\tau \in {}^{2\,\mathrm{lh}(\pi_0(p(x)))}({}^{\leq \mathrm{lh}(\pi_0(p(x)))}[\mathbb{N}])$

with $q \leq_{\mathbb{P}(B,\mathcal{I}\restriction B)} p \restriction L_{\mathrm{md}(p)}$ such that $q \Vdash_{\mathbb{P}(B,\mathcal{I}\restriction B)} \check{\tau} \subseteq \pi_1(p(\mathrm{md}(p)))$. Then by the induction hypothesis there is $q' \leq_{\mathbb{P}(B,\mathcal{I}\restriction B)} q$ satisfying the requirement. So $q' \cup \{(\mathrm{md}(p), p(\mathrm{md}(p)))\}$ is as desired.

Let $\mathcal{A}' \subseteq \mathbb{P}(L,\mathcal{I})$ be uncountable, have all elements in this dense set, and contain an element below every element of $\mathcal{A}$. It is enough to show that $\mathcal{A}'$ has an uncountable subset with all conditions compatible.

By the Δ-system lemma we can find an uncountable subset $\mathcal{A}''$ of $\mathcal{A}'$ such that $\{\mathrm{dom}(p) : p \in \mathcal{A}''\}$ is a Δ system with root r.

For each $x \in r$ there are countably many choices for $\pi_0(p(x))$, and then countably many choices for $\pi_1(p(x)) \restriction 2\,\mathrm{lh}(\pi_0(p(x))) \in {}^{2\,\mathrm{lh}(\pi_0(p(x)))}({}^{\leq \mathrm{lh}(\pi_0(p(x)))}[\mathbb{N}])$. So we can find an uncountable subset $\mathcal{A}'''$ of $\mathcal{A}''$ where all these choices are the same.

Now for all $p_0, p_1 \in \mathcal{A}'''$ we see they are compatible by constructing a q below both of them. q will have $\mathrm{dom}(q) = \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$ and is constructed by recursion along its domain using the way p_0 and p_1 are constructed, and if the next value $q(x)$ to be constructed has $x \in r$ take $q(x)$ to be $(\tau_{p_0 \restriction (L_x \cup \{x\})}, \pi_1(p_0(x)) \cup \pi_1(p_1(x)))$. □

Lemma III.54 (analogue of [7, Lemma 1.6]). *For all templates $(L,\mathcal{I})$, every $\dot{r}$ a $\mathbb{P}(L,\mathcal{I})$ name for a real, every $\dot{\varphi}$ a $\mathbb{P}(L,\mathcal{I})$ name for an element of ${}^{\mathbb{N}}({}^{\leq n}[\mathbb{N}])$ with $n \in \mathbb{N}$, and every $p \in \mathbb{P}(L,\mathcal{I})$, there are countable sets $A_r, A_\phi, A_p \subseteq L$ such that $\dot{r}$ is a $\mathbb{P}(A_r,\mathcal{I} \restriction A_r)$ name for a real, $\dot{\varphi}$ is a $\mathbb{P}(A_\phi,\mathcal{I} \restriction A_\phi)$ name for an element of ${}^{\mathbb{N}}({}^{\leq n}[\mathbb{N}])$, and $p \in \mathbb{P}(A_p,\mathcal{I} \restriction A_p)$.*

Proof. This is easy to see by induction on the rank of L and the facts that both $\dot{r}$ and $\dot{\varphi}$ are determined by countably many conditions in a $\mathbb{P}(B,\mathcal{I} \restriction B)$ for $B \subsetneq L$, and $p(\mathrm{md}(p))$ is a pair $(\sigma, \dot{\varphi})$. □

The following lemma, which is immediate from the definition of the posets and the completeness of embeddings lemma, will be used later to see that we indeed cofinally often take an extension by localization forcing.

Lemma III.55 (Embedding Localization Forcing, analogue of [2, Cor. 1.5])**.** *For any template* $(L, \mathcal{I})$*,* $x \in L$*, and* $A \subseteq L_x$ *such that* $A \in \mathcal{I}$*, we have that* $\mathbb{P}(A, \mathcal{I} \upharpoonright A) <\!\circ\, \mathbb{P}(A \cup \{x\}, \mathcal{I} \upharpoonright (A \cup \{x\}) \cong \mathbb{P}(A, \mathcal{I} \upharpoonright A) * \mathbb{LOC} <\!\circ\, \mathbb{P}(L, \mathcal{I})$*.* □

The following notion is very important to us; it will be used below to see that posets defined using different templates are isomorphic.

Definition III.56. For $(L, \mathcal{I})$ and $(L, \mathcal{I}')$ two templates, we say $\mathcal{I}$ is an *innocuous extension* of $\mathcal{I}'$ if $\mathcal{I}' \subseteq \mathcal{I}$ and for all $B \in \mathcal{I}$ with $B \subseteq L_x$ and all countable $A \subseteq B$ there is a $C \in \mathcal{I}'$ with $C \subseteq L_x$ such that $A \subseteq C$.

For this notion we have the following.

Lemma III.57 (Innocuous extensions, [2, Lemma 1.7])**.** *If* $(L, \mathcal{I})$ *and* $(L, \mathcal{I}')$ *are two templates with* $\mathcal{I}$ *an innocuous extension of* $\mathcal{I}'$*, then* $\mathbb{P}(L, \mathcal{I}) \cong \mathbb{P}(L, \mathcal{I}')$*.* □

The Template

The template we describe in this section is due to Shelah [31]; we work with the description of it in Brendle [7].

The template we use is $(L, \mathcal{I}) = (L(\mu, \lambda$ $), \mathcal{I}(\mu, \lambda$ $))$ with μ and λ

- $x \subset y$ and $y(\mathrm{lh}(x)) \in \lambda$ (elements get bigger by extending in λ), or
- $y \subset x$ and $x(\mathrm{lh}(y)) \in \lambda^*$ (elements get smaller by extending in λ^*), or
- if $n := \min\{m : x(m) \neq y(m)\}$ then $x(n) < y(n)$.

We say $x \in L$ is *relevant* if

- $\mathrm{lh}(x) \geq 3$ and is odd, and
- $x(n)$ is negative for odd n and positive for even n, and
- $x(\mathrm{lh}(x) - 1) < \omega_1$, and
- if $n < m$ are such that $x(n), x(m) < \omega_1$ and are even, then there are $\beta < \alpha$ such that $x(n-1) \in S^\alpha$ and $x(m-1) \in S^\beta$.

For a relevant x we define J_x to be the interval $[x \restriction (\mathrm{lh}(x) - 1), x)$.

Now we define $\mathcal{I}$ to be generated by taking finite unions of singletons of members of L, L_α for $\alpha \in \mu \cup \{\mu\}$, and J_x for relevant x.

Lemma III.58 ([7, Lemma 3.2])**.** *$(L, \mathcal{I})$ is a template.*

The Result

Now we are ready to put all our work together to prove the following theorem which clearly implies Theorem III.40.

Theorem III.59. *Let $\lambda > \mu > \aleph_1$ be regular cardinals with $\lambda = \lambda^\omega$. Then $\mathbb{P}(L, \mathcal{I}) \Vdash$ "$\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mu$ and $\mathfrak{a}_g = \lambda = 2^{\aleph_0}$".*

Since μ (identified with sequences of length 1) is cofinal in L, and $L_\mu \in I$, the slaloms added at the coordinates associated to μ localize all reals in the generic extension as follows. Let G be $\mathbb{P}(L, \mathcal{I})$ generic and r a real in $V[G]$. Then by Lemma

III.54 we find a countable $A \subseteq L$ such that $r \in V[G \cap \mathbb{P}(A, \mathcal{I} \restriction A)]$. There is an $\alpha \in \mu$ such that $A \subseteq L_\alpha$, so that the real r is in $V[G \cap \mathbb{P}(L_\alpha, \mathcal{I} \restriction L_\alpha)]$. Then Lemma III.55 shows that r is localized by the slalom added at coordinate α.

Since for a family of reals $\{r_\gamma : \gamma < \beta\}$ with $\beta < \mu$ we can do similar reasoning, we see that these slaloms actually capture all families of reals of size less than μ in the generic extension as follows. Code the family $\{r_\gamma : \gamma < \beta\}$ into a function $\dot{F} : \beta \times \omega \to \mathbb{N}$. This name is determined by $\beta \times \omega$ many antichains $\{B_{\theta,n} : (\theta, n) \in \beta \times \omega\}$, where the antichain $B_{\theta,n}$ consists of conditions deciding the value of $\dot{F}(\theta, n)$. By the c.c.c. all these antichains are countable; enumerate for each θ and n the antichain $A_{\theta,n}$ by $\{p_{\theta,n,k} : k \in \mathbb{N}\}$. For each $p_{\theta,n,k}$ there exists a countable set $A_{\theta,n,k}$ such that $p_{\theta,n,k} \in \mathbb{P}(A_{\theta,n,k}, \mathcal{I} \restriction A_{\theta,n,k})$ (by Lemma III.54). Since these are $< \mu$ many countable sets, there is an $\alpha < \mu$ such that all are contained in L_α. This means $\dot{F}$, and therefore the family $\{r_\gamma : \gamma < \beta\}$, are in $V^{\mathbb{P}(L_\alpha, \mathcal{I} \restriction L_\alpha)}$. The slalom added at coordinate α then localizes all of them.

These two arguments together show that $\mathrm{cof}(\mathcal{N}) = \mathrm{add}(\mathcal{N}) = \mu$ in the generic extension.

By counting names for reals we see that in the forcing extension $2^{\aleph_0} \leq \lambda$ as follows. A nice name for a real is determined by countably many maximal antichains and countably many reals. This gives us

$$(\#\text{maximal antichains})^{\aleph_0} \cdot 2^{\aleph_0}$$

as an upper bound for the number of nice names for reals. We have to show that the number of maximal antichains is bounded by λ. Because the forcing satisfies Knaster's condition, all antichains are countable. This gives us

$$|\mathbb{P}(L, \mathcal{I})|^{\aleph_0 \times \aleph_0} \cdot 2^{\aleph_0}$$

as an upper bound for the number of nice names for reals. We show by induction on the rank of L that $|\mathbb{P}(L,\mathcal{I})| \leq \lambda$. Assume that for all B of rank less than L we have that $|\mathbb{P}(B,\mathcal{I} \restriction B)| \leq \lambda$. Then $|\mathbb{P}(L,\mathcal{I})| \leq \Sigma_{B\in\mathcal{I},B\neq L}|\mathbb{P}(B,\mathcal{I} \restriction B)|^{\aleph_0} \cdot (2^{\aleph_0} \cdot \aleph_0) = \Sigma_{B\in\mathcal{I},B\neq L}\lambda^{\aleph_0} \cdot (2^{\aleph_0} \cdot \aleph_0) = |\mathcal{I}| \cdot \lambda = \lambda$ (using the induction hypothesis, the fact that $\lambda^{\aleph_0} = \lambda$, and the continuum hypothesis).

Now we will show by an isomorphism-of-names argument that in the forcing extension no cofinitary group of cardinality less than $2^{\aleph_0}$ and bigger than or equal to μ can exist (remember that $\mathfrak{a}_g \geq \mathrm{non}(\mathcal{M})$, as was mentioned above, and that $\mathrm{add}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$) which completes the proof.

Let $\dot{G}$ be a name for a cofinitary group of size $< \lambda$ and $\geq \mu$, say of size κ. We can find names $\dot{g}_\alpha$ ($\alpha < \kappa$) such that in the forcing extension, $\{\dot{g}_\alpha : \alpha < \kappa\}$ is this cofinitary group. For each $\dot{g}_\alpha$ we find maximal antichains $\{p^\alpha_{n,k,i} : k, i \in \mathbb{N}\}$ (for each $n \in \mathbb{N}$) in $\mathbb{P}(L,\mathcal{I})$ such that $p^\alpha_{n,k,i} \Vdash \dot{g}_\alpha(n) = k$. Let $B^\alpha = \cup\{\mathrm{dom}(p^\alpha_{n,k,i}) : n, k, i \in \mathbb{N}\}$. Each B^α is a countable set of finite sequences from $\lambda^* \cup \lambda$. If we close it under initial segments with respect to $\subseteq$ (that is not with respect to the order given on L) we can assume it to be a tree (in the following, "tree" will always mean tree with respect to $\subseteq$).

We will analyze countable subtrees of L for a moment to see how we can use these trees.

So let $T_0, T_1 \subseteq L$ be countable subtrees of L. We say $T_0 \cong T_1$ if there is a bijection $\varphi : T_0 \to T_1$ such that

- φ respects the order of L: if $x < y$ then $\varphi(x) < \varphi(y)$;
- φ is a tree map: $\varphi(x \restriction n) = \varphi(x) \restriction n$ and $\mathrm{lh}(\varphi(x)) = \mathrm{lh}(x)$;
- φ respects the structure on the trees used to determine relevance: $\varphi(x)(n)$ is positive, less than ω_1, or a member of S^α iff $x(n)$ is. (One could choose to

have a somewhat less restrictive requirement here, but this one suffices for our purposes.)

If $T_0 \cong T_1$ as witnessed by φ, then φ induces an isomorphism of $\mathcal{I} \upharpoonright T_0$ with $\mathcal{I} \upharpoonright T_1$.

The number of types of trees can now be determined as follows. The number of trees up to equivalence for maps satisfying the first two items is less than the number of countable subtrees of ${}^{<\omega}(\omega_1^* \cup \omega_1)$. This gives us an upper bound of $\aleph_1^{\aleph_0} = 2^{\aleph_0}$. For each tree the information for the last item can be encoded in a map from the set of nodes, size $\aleph_0$, to $\omega_1 \times \{0,1\} \times \{0,1\}$ (ω_1 for the labels of the S^α and the two $\{0,1\}$ as labels for positive vs. negative, and for positive nodes bigger vs. smaller than ω_1). There is a total of $\aleph_1^{\aleph_0} = 2^{\aleph_0}$ of these maps, which gives us a grand total of $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$ isomorphism types of trees.

If we have an isomorphism of linear orders that induces an isomorphism on the associated templates, then this isomorphism induces an isomorphism of the associated posets. So from the above we can conclude that if $T_0 \cong T_1$ then $\mathbb{P} \upharpoonright T_0 \cong \mathbb{P} \upharpoonright T_1$.

Brendle in the proof of Theorem 3.3, [7, pp. 21–23], shows how in the set $\{B^\alpha : \alpha < \kappa\}$ we can find an ω_1 size subset, which after renumbering we can assume to be $\{B^\alpha : \alpha < \omega_1\}$, and a B^κ such that:

- there is a coherent set of maps $\phi_{\alpha,\beta}$ ($\alpha, \beta < \omega_1$) and $\phi_{\alpha,\kappa}$ ($\alpha < \omega_1$) such that the $\phi_{\alpha,\beta} : B^\alpha \to B^\beta$ are isomorphisms of trees, and $\phi_{\alpha,\kappa} : B^\alpha \to B^\kappa$ is order preserving;
- $\phi_{\alpha,\beta}(p^\alpha_{n,k,i}) = p^\beta_{n,k,i}$ (names map to names);
- for any $\beta < \kappa$, $\mathcal{I} \upharpoonright B^\kappa \cup B^\beta$ is an innocuous extension of the image of $\mathcal{I} \upharpoonright B^\alpha \cup B^\beta$ for some $\alpha < \omega_1$ (the image under the mapping induced by the mappings ϕ).

In fact it is clear from his construction that for any $\beta_0, \ldots, \beta_i < \kappa$, $\mathcal{I} \restriction B^\kappa \cup (\bigcup_{j \leq i} B^{\beta_j})$ is an innocuous extension of the image of $\mathcal{I} \restriction B^\alpha \cup (\bigcup_{j \leq i} B^{\beta_j})$ for some $\alpha < \omega_1$.

If we define $\dot{g}_\kappa$ to be the name for a bijection by $p^\kappa_{n,k,i} = \phi_{\alpha,\kappa}(p^\alpha_{n,k,i})$, we get a name such that in the generic extension $\{\dot{g}_\alpha : \alpha < \kappa\} \cup \{\dot{g}_\kappa\}$ is a cofinitary group that properly includes $\{\dot{g}_\alpha : \alpha < \kappa\}$, so the group we started with was not maximal: Let $\mathcal{G}$ be $\mathbb{P}(L, \mathcal{I})$ generic, and in the generic extension, $w(x) \in W_{\dot{G}}$. We need to see that $w(\dot{g}_\kappa)$ is cofinitary. Let $\dot{g}_{\beta_j}$ with $j < i$ be the elements of $\dot{G}$ appearing in w. Then $V[\mathcal{G}] \models$ "$w(\dot{g}_\kappa)$ is cofinitary", iff $V[\mathcal{G} \cap \mathbb{P}(B^\kappa \cup (\bigcup_{j<i} B^{\beta_j}), \mathcal{I} \restriction B^\kappa \cup (\bigcup_{j<i} B^{\beta_j})) \models$ "$w(\dot{g}_\kappa)$ is cofinitary". But by the above $\mathbb{P}(B^\kappa \cup (\bigcup_{j<i} B^{\beta_j}), \mathcal{I} \restriction B^\kappa \cup (\bigcup_{j<i} B^{\beta_j})) \cong \mathbb{P}(B^\alpha \cup (\bigcup_{j<i} B^{\beta_j}), \mathcal{I} \restriction B^\alpha \cup (\bigcup_{j<i} B^{\beta_j}))$ for some $\alpha < \omega_1$ fixing the B^{β_j} (therefore fixing the $\dot{g}_{\beta_j}$ and mapping $\dot{g}_\kappa$ to $\dot{g}_\alpha$). This means that $V[\mathcal{G} \cap \mathbb{P}(B^\kappa \cup (\bigcup_{j<i} B^{\beta_j}), \mathcal{I} \restriction B^\kappa \cup (\bigcup_{j<i} B^{\beta_j}))] \models$ "$w(\dot{g}_\kappa)$ is cofinitary" if $V[\mathcal{G} \cap \mathbb{P}(B^\alpha \cup (\bigcup_{j<i} B^{\beta_j}), \mathcal{I} \restriction B^\alpha \cup (\bigcup_{j<i} B^{\beta_j}))] \models$ "$w(\dot{g}_\alpha)$ is cofinitary", which is true since $\dot{G}$ is forced to be a cofinitary group.

3.5 Definability

3.5.1 There Does Not Exist a K_σ Maximal Cofinitary Group

A set is K_σ if it is a countable union of compact sets; every K_σ set is eventually bounded in the following sense.

Definition III.60. (i). We write $\forall^* n\, \varphi(n)$ if for all but finitely many $n \in \mathbb{N}$, $\varphi(n)$.

(ii). For $f, g \in {}^{\mathbb{N}}\mathbb{N}$, f is *eventually bounded* by g, written $f <^* g$, if $\forall^* n\, f(n) < g(n)$ (if there exists $k \in \mathbb{N}$ such that for all $l > k$, $f(l) < g(l)$.

(iii). A set $S \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *eventually bounded* if there exists $f \in {}^{\mathbb{N}}\mathbb{N}$ such that for all $g \in S$, $g <^* f$.

Theorem III.61. *If G is a cofinitary group that is eventually bounded, then G is not maximal.*

The basic idea in this proof is to use the bound to get an interval partition which can be used similarly to how the orbits were used in the proof of Theorem III.11.

Proof. Let G be a cofinitary group that is eventually bounded. This means G is contained in a set of the form $\{g \in {}^{\mathbb{N}}\mathbb{N} : g <^* f\}$ where we can assume $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function with $f(0) > 0$. We will use this bound f to construct an interval partition, with a distinguished point in each interval, that we in turn use to construct $h \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, h\rangle$ is cofinitary.

Define $I = \big\langle \big([i_n, i_{n+1}), p_n\big) : n \in \mathbb{N}\big\rangle$ by $i_0 := 0$, $p_n := f(i_n)$, and $i_{n+1} := f(p_n)$. The main property of this sequence is

$$\forall g \in G\ \forall^* n\ g(p_n) \in [i_n, i_{n+1}),$$

which follows easily from the fact that all elements of G are nearly everywhere strictly bounded by f.

Define h by finite approximations: let $h_0 := \emptyset$. Then at step s define h_{s+1} from h_s as follows:

1. Let $a := \min(\mathbb{N} \setminus \mathrm{dom}(h_s))$, let n be the least number such that for all $l \geq n$ we have $[i_l, i_{l+1}) \cap \big(\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s) \cup \{a\}\big) = \emptyset$, and set $\bar{h}_{s+1} := h_s \cup \{(a, p_n)\}$.

2. Let $b := \min(\mathbb{N} \setminus \mathrm{ran}(\bar{h}_{s+1}))$, let m be the least number such that for all $l \geq m$ we have $[i_l, i_{l+1}) \cap \big(\mathrm{dom}(\bar{h}_{s+1}) \cup \mathrm{ran}(\bar{h}_{s+1}) \cup \{b\}\big) = \emptyset$, and set $h_{s+1} := \bar{h}_{s+1} \cup \{(p_m, b)\}$.

Note that the a and b used in this construction satisfy $a \leq b \leq a + 1$. To see this first note that the p_n used go alternately into the domain and range, starting with the range. Then by induction it follows that if $a = b$ in an iteration then the least number in $\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s) \setminus a$ is in the range. If $a+1 = b$, then that least number is in the domain. The note quickly follows from these facts.

The main properties of h are the following (for $n > 0$).

1. If $a \in [i_n, i_{n+1}) \setminus \{p_n\}$ and $(a, b) \in h$, then $b > i_{n+1}$.

2. If $b \in [i_n, i_{n+1}) \setminus \{p_n\}$ and $(a, b) \in h$, then $a > i_{n+1}$.

3. If $(a, p_n), (p_n, b) \in h$, then at most one of a and b is less than i_n.

4. If $(a_0, b_0), (a_1, b_1) \in h \cup h^{-1}$ and $a_0 < a_1 < b_0$, then $b_1 < a_0$ or $b_1 > b_0$.

The first three of these follow from the observation that for any pair added to h one of the coordinates is a p_n and this p_n is from a later interval than the other coordinate is in (sometimes both coordinates are equal to p_n for some n, but only one is used as such in the construction). The last one follows from the fact that any added pair has one coordinate strictly bigger than any number mentioned before and the note on the order of a and b above.

Taking the first three properties of h together we get that for any $n > 0$ there is at most one pair in $h \cup h^{-1}$ with one coordinate in $[i_n, i_{n+1})$ and the other smaller than i_n. From this we see that if $l < i_n$ and $h\ (l) \in [i_n, i_{n+1})$ for $\epsilon \in \{-1, +1\}$ then $h\ (l) = p_n$. Moreover for $m \in [i_n, i_{n+1}) \setminus \{p_n\}$ both $h(m)$ and $h^{-1}(m)$ are bigger than i_{n+1}; this is also the case for $h\ (p_n)$, but not for $h^{-}\ (p_n) = l$.

Now we show that $\langle G, h \rangle$ is cofinitary. Let us assume, towards a contradiction, that $w(x) = g_0 x^{k_0} g_1 \cdots g_k x^{k_m} g_{m+1} \in W_G$ is such that $w(h)$ has infinitely many fixed points. We can also assume that $g_{m+1} = \mathrm{Id}$, since this only requires conjugation by g_{m+1} and this does not change the number of fixed points.

We normalize the word w further. For this we work above M, the least number such that for all $n \geq M$ and all $g \in G$ appearing in w we have $g(n) < f(n)$. We want a conjugate w' of w of the form $g_l x^{k_l} g_{l+1} \cdots\ g_{m+1} g_0 x^{k_0} \cdots x^{k_{l-1}}$ such that for infinitely many of its fixed points, n, the image after the first application of h (if $k_{l-1} > 0$) or h^{-1} (if $k_{l-1} < 0$) is bigger than n. Such a conjugate w' exists if for infinitely many fixed points we can find a location in the evaluation path where an application of h increases the number. So suppose that you can't do this; then for all but finitely many fixed points every application of h leads to a smaller number. In this case we can find an n, a fixed point of $w(h)$ such that no point in its evaluation path $\bar{z}$ is less than M and for all i such that $w_i = h$, $\epsilon \in \{+1, -1\}$, we have $z_{i+1} < z_i$ (remember $z_{i+1} = w_i z_i$).

Now since we start in $w(h)$ by applying h we get $z_1 < z_0 = n$. After this we cannot get back to z_0 as any application of a g appearing in w to a number less than or equal to z_1 will lead to a number strictly less than z_0 (z_0 is in the middle of an interval which does not contain z_1 and z_0 is the f image of the start of the interval it is in). And any application of h to a number strictly less than z_0 will lead to a number strictly less than z_1 (follows from the assumption and 4). This contradiction shows that a conjugate w' as desired exists.

We will study this conjugate w' of w; if it can't have infinitely many fixed points neither can w. There are only finitely many points whose evaluation path in $w'(h)$ involves natural numbers less than M. Leave these out of consideration.

Let $\bar{z}$ be the evaluation path of $w'(h)$ on n, a fixed point for this word where the image after the first application of h is bigger than n. There is a least m such that there is an $a \in \mathbb{N}$ such that $z_{m+1} < i_a \leq z_m$.

If for some l we have $z_{l+1} > z_l$ by an application of h (either h or h^{-1}) we have

$z_{l+1} = p_b$ for some $b \in \mathbb{N}$. If we now apply h again (the same of h or h^{-1}) we map to a p_m with $p_m > p_b$. So if we are in w' at some x^l, repeatedly applying h, once we start increasing we will keep on increasing.

If after such applications of h where we increase we apply a $g \in G$ as indicated by w', then g doesn't map the element out of the interval it is in (we are working above M' where no element of g appearing in w' can map further than f or f^{-1}).

Now we know that $z_m = p_k$ for some k, z_{m+1} is obtained from z_m by an application of h, z_m is obtained from z_{m-1} by an application of some $g \in G$ and z_{m-1} is obtained from z_{m-2} by an application of h which was increasing. From the last fact in the last sentence we know $z_{m-1} = p_l$ for some l. Since p_l and p_k are in the same interval, $p_l = p_k$ and we have found a fixed point for this $g \in G$.

So we have found from a fixed point of $w'(h)$ a fixed point for some $g \in G$ appearing in w'. Also, any fixed point of a $g \in G$ appearing in w' can only be used in the evaluation path of finitely many points (and only in the evaluation path of one fixed point if g only appears once in w'). From this we see that if $w'(h)$ has infinitely many fixed points, so does some $g \in G$. This is the contradiction we were looking for. □

3.5.2 A Coanalytic Maximal Cofinitary Group

In this subsection we will prove the following theorem.

Theorem III.62. *The axiom of constructibility implies that there exists a coanalytic maximal cofinitary group.*

The proof will be very much related to the proof of the analogous result for very mad families, which is in Section 2.5, but there are some essential differences.

The construction of very mad families (and many other types of maximal almost disjoint families) proceeds by adding one new member at a time. We recursively construct the family to be $\mathcal{A} = \{f_\alpha : \alpha < \omega_1\}$. Then under the axiom of constructibility we have to prove a coding lemma of the following form.

Lemma III.63 (Coding Lemma — Generic Form)**.** *If A is a countable almost disjoint family and $z \in 2^{\mathbb{N}}$, we can construct a new member f to adjoin to the family such that*

(i). z is recursive in f, and

(ii). if we iterate the construction ω_1 many times we construct a maximal almost disjoint family.

In fact the construction has to be such that z is uniformly recursive in f; the function computing z from f should not depend on A or on other parameters in the construction.

If this can be achieved, the proof as in Section 2.5 can be easily adjusted to yield a coanalytic maximal almost disjoint family, for whichever notion of almost disjoint you are considering.

Using this method Su Gao and Yi Zhang were able to prove the following in [12].

Theorem III.64. *The axiom of constructibility implies that there exists a maximal cofinitary group with a coanalytic generating set.*

They prove a nice version of the generic type coding lemma, and the generating set is constructed in the right way for the general method to apply.

The difficulty in showing that the whole group can be coanalytic is that when you add a new generator you also add countably many other new elements. In the construction we will use the method of good extensions, which means that the new generator will be free over all that came before. Then for all $w \in W_G \setminus G$ we will have $w(g) \notin G$. And all these elements need to encode z for the method to work.

The following lemma shows that in the case of cofinitary groups we cannot get z uniformly recursive in f in the coding lemma, when our construction is computable. This does not prove that uniform computability is not possible as the construction does not need to be computable, and moreover, it only has to work over a fixed group (the one constructed in previous steps of the construction).

Proposition III.65. *There do not exist recursive functionals $\Psi(X, Z, n)$ and $\Phi(X, n)$ such that for all countable cofinitary groups G, and all $z \in 2^{\mathbb{N}}$ the function $g \in \mathrm{Sym}(\mathbb{N})$ defined by $g(n) = \Psi(G, z, n)$ satisfies that $\langle G, g\rangle$ is cofinitary, and for all $w \in W_G$ we have that $z(n) = \Phi(w(g), n)$.*

Proof. Let G be given to us as a countable sequence $\langle g_i : i \in \mathbb{N}\rangle$, and assume that Ψ and Φ as in the statement do exist.

Pick a countable cofinitary group G, and a $z \in 2^{\mathbb{N}}$ with $z(0) = 0$. Define g from G and z using Ψ as in the statement of the lemma. Let $u = \mathsf{use}(\Phi, g, 0)$, the use of g by the functional Φ when calculating $\Phi(g, 0)$. Then for all $h \in \mathrm{Sym}(\mathbb{N})$, if $h \restriction u = g \restriction u$ then $\Phi(h, 0) = 0$.

Let $z' \in 2^{\mathbb{N}}$ be such that $z'(0) = 1$. Define g' from G and z' using Ψ as in the statement of the lemma. Let $U = \mathsf{use}(\Psi, G, z', g' \restriction u)$, the maximal use of G and z by Ψ in calculating $g'(0), \ldots, g'(u-1)$.

So in determining $g'(0), \ldots, g'(u-1)$ no use is made of any group element in the enumeration $\langle g_i : i \in \mathbb{N}\rangle$ with index $i > U$.

Now pick a new cofinitary group $\bar{G}$ and enumeration of it $\langle \bar{g}_i : i \in \mathbb{N}\rangle$ such that $\bar{g}_i = g_i$ for $i \leq U$ and there are elements g_l and g_k such that $g_l(g' \restriction u)g_k = g \restriction u$.

Define g'' from $\bar{G}$ and z' using Ψ as in the statement of the lemma. Then $g'' \restriction$

$u = g' \restriction u$. However if we choose $w(x) = g_l x g_k$, then $w(g'') = g \restriction u$, which means that $\Phi(w(g''), 0) = 0$ contradicting the fact that Φ computes z' from $w(g'')$. $\square$

Now that we know what the difficulty is, we will show how to deal with it. We will recursively construct the maximal cofinitary group. To make the coding work out though we have to start with a specific countable group.

Let G_0 be the countable cofinitary group generated by h defined as follows:

$$h(n) = \begin{cases} n-2, & \text{if } n \text{ is even and not zero;} \\ n+2, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n = 0. \end{cases}$$

Then there is a formula only involving natural number quantifiers $\phi_{G_0}(x)$ that defines this group as a subgroup of $\mathrm{Sym}(\mathbb{N})$.

The coding method we use has two cases and a parameter. But with these it will be uniform; there exists a recursive functional $\Phi(X, m, \epsilon, n)$ such that if z is encoded in f we have that there exist $m \in \mathbb{N}$ and $\epsilon \in \{0, 1\}$ such that for all $n \in \mathbb{N}$ we have $z(n) = \Phi(f, m, \epsilon, n)$.

The encoding will be as follows; z is encoded in f with parameter $(m, 0)$ iff

$$(k_n, z(n)) = f^n(m), \text{ for some } k_n \in \mathbb{N};$$

z is encoded in f with parameter $(m, 1)$ iff

$$(k_n, z(n)) = f(hf)^n(m).$$

This encoding will be done in the following way. At some point in the construction, we have already constructed a finite approximation p to the new generator g. We then start encoding into a new word $w \in W_G$. Let $w = g_0 x^{k_0} g_1 \cdots x^{k_l} g_{l+1}$ with $g_i \in G$

$(i \leq l+1)$ and $k_i \in \mathbb{Z} \setminus \{0\}$. Pick m such that $g_{l+1}(m) \notin \mathrm{dom}(p) \cup \mathrm{ran}(p)$, and let $\gamma = 0$ if w does not have a proper conjugate subword, $\gamma = 1$ otherwise. We extend p by taking good extension with respect to certain words, extending the evaluation path of $w(p)$ for m. We do this until $a = (w \upharpoonright \mathrm{lh}(w) - 2)(m)$ is defined. Then (assuming $k_0 > 0$, the other case is analogous) we choose a b such that $p \cup \{(a, b)\}$ is a good extension with respect to certain words, such that $w(p \cup \{(a, b)\})(m) \in \{(k, z(0)) : k \in \mathbb{N}\}$, and such that we can encode $z(1)$ into the next location.

This last requirement is where the two different types of encoding play a role. If w has no proper conjugate subword, then since G is cofinitary there is only finitely much restriction from the requirement that $g_{l+1}(g_0(b)) \notin \mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \{b\}$. If w does have a proper conjugate subword, then we will always have that $g_{l+1}(g_0(b)) = b$. This is why in that case we "twist" by h. The next location we then want to encode in is $h(g_0(b))$ and, again since G is cofinitary, we will have only finitely much restriction from requiring $g_{l+1}(h(g_0(b))) \notin \mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \{b\}$.

With this we have enough information to state and prove the coding lemma for cofinitary groups.

Lemma III.66. *Let G be a countable cofinitary group, $F \leq \mathrm{Sym}(\mathbb{N}) \setminus G$ a countable family of permutations such that for all $f \in F$ the group $\langle G, f \rangle$ is cofinitary, and $z \in 2^{\mathbb{N}}$. Then there exists g such that $\langle G, g \rangle$ is cofinitary, for all $f \in F$ the set $f \cap g$ is infinite, and z is recursive in $w(g)$ for all $w \in W_G \setminus G$.*

Proof. $W_G \setminus G$ is countable, enumerate it by $\langle w_n : n \in \mathbb{N} \rangle$, and enumerate F by $\langle f_n : n \in \mathbb{N} \rangle$.

Start by setting $g := \emptyset$, $A := \emptyset$ and $\langle c_n : n \in \mathbb{N} \rangle$ with all $c_n := \emptyset$. g will be the permutation we construct, so at any time it will be a finite injective function. A is a set of numbers; it is the set of numbers in domain and range that are being used in coding. We have to avoid this set in all steps other than coding steps . It will always be finite and any number will stay in it for only finitely many stages of the construction. $\langle c_n : n \in \mathbb{N}\rangle$ is a sequence of which at any time an initial segment will contain triples that hold information on how far we are in the coding, how we are coding, and where the coding currently is being done.

At step $s \in \mathbb{N}$ in the construction we do the following:

- Extend Domain: Set $a := \min\{\mathbb{N} \setminus (\text{dom}(g) \cup A)\}$. By the domain extension lemma for all but finitely many b the extension $g \cup \{(a, b)\}$ is a good extension of g with respect to all words w_i, $i \leq s$. Choose b to be the least such number such that $b \notin A$ and set $g = g \cup \{(a, b)\}$.

- Extend Range: Set $b := \min\{\mathbb{N} \setminus (\text{ran}(g) \cup A)\}$. By the range extension lemma for all but finitely many a the extension $g \cup \{(a, b)\}$ is a good extension of g with respect to all words w_i, $i \leq s$. Choose a to be the least such number such that $b \notin A$ and set $g = g \cup \{(b, a)\}$.

Note: these two sub-steps ensure that g will be a permutation of $\mathbb{N}$; no number stays in A long enough to cause problems.

- Hit f: For each $j \leq s$ in turn do the following:

  By the Hitting f lemma, for all but finitely many a the extension $g \cup \{(a, f_j(a))\}$ is a good extension of g with respect to all words w_i, $i \leq s$. Choose a to be the least such number such that $a, f_j(a) \notin A$ and set $g = g \cup \{(a, f_j(a))\}$.

Note: this ensures for all $f \in F$ that $f \cap g$ is infinite.

- Coding: For each $j < s$ in turn do the following:

  c_j is a triple (m, l, γ), where m denotes where the coding is taking place, l denotes the next location of z to encode, and γ determines how to encode.

Let n be the largest number such that $a := (w_j \restriction n)(g)(m)$ is defined. Then $w_j = w' g_j x^k x^\delta (w_j \restriction n)$, where $w' \in W_G$, $g_j \in G$, and $k \geq 0$ if $\delta = 1$ and $k \leq 0$ if $\delta = -1$.

<u>Case $\delta = 1$:</u>

By the domain extension lemma, for all but finitely many b the extension $g \cup \{(a, b)\}$ is a good extension of g with respect to all words w_i, $i \leq s$.

<u>Subcase $k > 0$:</u>

Choose b to be the least number such that $b \notin A \cup \mathrm{dom}(p)$, set $g = g \cup \{(a, b)\}$ and replace a in A by b (so a is no longer a member of A but b now is).

<u>Subcase $k = 0$:</u>

<u>SubSubcase $w' = w'' x^{\delta'}$ ($\delta' \in \{-1, 1\}$):</u>

Choose b to be the least number such that $b \notin A$ and $g_j(b) \notin A \cup \mathrm{dom}(g) \cup \mathrm{ran}(g)$ (in fact depending on δ' we only care about avoiding one of $\mathrm{dom}(g)$ or $\mathrm{ran}(g)$). Set $g = g \cup \{(a, b)\}$ and replace a in A by $g_j(b)$.

<u>SubSubcase $w' = \emptyset$:</u> (This is where the actual coding happens.)

Choose b to be the least number such that $b \notin A$, $g_j(b) \notin A$, $g(b) \in \{(c, z(l)) : c \in \mathbb{N}\}$ and if $\gamma = 0$ $w_0(g_i(b)) \notin A \cup \mathrm{dom}(g) \cup \mathrm{ran}(g) \cup \{b\}$ or if $\gamma = 1$ then $w_0(h(g_i(b))) \notin A \cup \mathrm{dom}(p) \cup \mathrm{ran}(g) \cup \{b\}$.

The requirements $b \notin A$, $g_j(b) \notin A$, and $w_0(g_j(b)) \notin A \cup \mathrm{dom}(p) \cup \mathrm{ran}(p)$ or $w_0(h(g_j(b))) \notin A \cup \mathrm{dom}(p) \cup \mathrm{ran}(p)$ exclude finitely many possibilities for b. Since G is cofinitary, $w_0(g_j(b)) \neq b$ or $w_0(h(g_j(b))) \neq b$ also excludes finitely many possibilities. So we can choose $b \in g_j^{-1}[\{(c, z(l)) : c \in \mathbb{N}\}]$ satisfying the last condition on b.

Then set $g = g \cup \{(a, b)\}$, replace a in A by $w_0(g_i(b))$ (if $\gamma = 0$) or $w_0(h(g_i(b)))$ (if $\gamma = 1$) and set $c_j := (g_j(b), l+1, 0)$ (if $\gamma = 0$) or $c_j := (h(g_j(b)), l+1, 1)$ (if $\gamma = 1$). (γ is set in Extending Coding below and explained on page 105.)

Case $\delta = -1$: The method and (sub)subcases are analogous to the case $\delta = 1$ but using the range extension lemma.

- Extending Coding: If w_s has a proper conjugate subword, set $\gamma = 1$; otherwise set $\gamma = 0$. Then let a be the least number such that if $w_s = w' g_s$, then $g_s(a) \notin \mathrm{dom}(g) \cup \mathrm{ran}(g) \cup A$. Add $g_s(a)$ to A and set $c_s = (a, 0, \gamma)$. This indicates that at the next stage we will start encoding $z(0)$ into location a for w_s.

Note: with the explanation before the lemma this ensures that the coding happens correctly. □

With the above indicated changes, Lemma II.24 has to be modified to be the following.

Lemma III.67.

$$g \in G \Leftrightarrow \phi_{G_0}(g) \vee \exists(m, \epsilon)\big[\ \textit{the model encoded in } g \textit{ is wellfounded} \wedge$$
$$\forall \langle E_\omega, r, u\rangle\ \varphi(\langle E_\omega, r, u\rangle, g) \wedge \chi(E_\omega, r) \rightarrow r(\ulcorner u \in \mathcal{A}\urcorner, \bar{\emptyset}) = 1\big].$$

This is clearly still a Π^1_1 formula, showing the result.

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ABSTRACT

Cofinitary Groups and Other Almost Disjoint Families of Reals

by

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We study two different types of (maximal) almost disjoint families: very mad families and (maximal) cofinitary groups.

For the very mad families we prove the basic existence results. We prove that MA implies there exist many pairwise orthogonal families, and that CH implies that for any very mad family there is one orthogonal to it. Finally we prove that the axiom of constructibility implies that there exists a coanalytic very mad family.

Cofinitary groups have a natural action on the natural numbers. We prove that a maximal cofinitary group cannot have infinitely many orbits under this action, but can have any combination of any finite number of finite orbits and any finite (but nonzero) number of infinite orbits.

We also prove that there exists a maximal cofinitary group into which each countable group embeds. This gives an example of a maximal cofinitary group that is not a free group. We start the investigation into which groups have cofinitary actions.

The main result there is that it is consistent that $\bigoplus_{\alpha\in\aleph_1}\mathbb{Z}_2$ has a cofinitary action.

Concerning the complexity of maximal cofinitary groups we prove that they cannot be K_σ, but that the axiom of constructibility implies that there exists a coanalytic maximal cofinitary group.

We prove that the least cardinality $\mathfrak{a}_g$ of a maximal cofinitary group can consistently be less than the cofinality of the symmetric group. Finally we prove that $\mathfrak{a}_g$ can consistently be bigger than all cardinals in Cichoń's diagram.