COMPARING NOTIONS OF RANDOMNESS

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ABSTRACT. It is an open problem in the area of effective (algorithmic) randomness whether Kolmogorov-Loveland randomness coincides with Martin-Löf randomness. Joe Miller and André Nies suggested some variations of Kolmogorov-Loveland randomness to approach this problem and to provide a partial solution. We show that their proposed notion of injective randomness is still weaker than Martin-Löf randomness. Since in its proof some of the ideas we use are clearer, we also show the weaker theorem that permutation randomness is weaker than Martin-Löf randomness.

1. INTRODUCTION

There are currently many competing notions of randomness, based on different intuitions of randomness. Some are based on the idea that no random real should belong to certain measure zero sets, others on the frequency interpretation of probability, and yet others on the notion of a fair betting game. Some of these notions are known to be equivalent, others are known to be not equivalent, and for yet others, it is not known whether they are equivalent. This paper is a contribution to this classification.

The notions of randomness we will be concerned with in this paper are all based on the notion of a martingale. A martingale is a formalization of the idea of a fair betting game; while betting on the outcome of a coin flip the game would be fair if the expected value of your capital after the game is the same as before the game. That means that your win on heads is the same as your loss on tails. A martingale then describes a game consisting of simple games like that infinitely often repeated. Part of the intuition for using martingales

to define randomness is that if the real is random you should not be able to predict the bits; this means that in the game against the real (considered as infinitely many games against bits) your capital will not be unbounded.

For the different notions we study, the main differences lie in the effectiveness of the martingale, the order in which the martingale bets on bits, and the speed by which the martingale is required to succeed. One of the big open questions in the area is whether the notions of Martin-Löf randomness (a notion of monotonic randomness with a very weakly effective martingale) and Kolmogorov-Loveland randomness (a notion of nonmonotonic randomness with a somewhat more effective, but not monotonic, martingale) coincide. Joe Miller and André Nies [MN06MN06, MN06] suggested a weakening of Kolmogorov-Loveland randomness as a way to approach this question. The weakening involves limiting the freedom of the nonmonotonic martingale in choosing the next bit to bet on.

We show that their notion of injective randomness does not coincide with Martin-Löf randomness.

We start in the next subsection with the definitions of the different types of martingales and some background. Then, in Section 2, we prove that permutation randomness does not coincide with Martin-Löf randomness. This is a weaker theorem than the theorem we show in Section 3. However, the proof for the permutation random case introduces many of the ideas in a simpler context.

1.1. **The Definitions and Background.** The space we are working in is the Cantor space 2^ω, the space of infinite sequences of zeros and ones, with the topology induced by the sets $[\sigma] = \{A \in 2^\omega \mid \sigma \subseteq A\}$ for any $\sigma \in 2^{<\omega} = \bigcup_{n \in \mathbb{N}} 2^n$, that is σ a finite sequence of zeros and ones. If $\sigma \in 2^{<\omega}$ then $\sigma \in 2^n$ for some n, we write $|\sigma|$ for this n. Note that when convenient we will use the convention that $n = \{0, \ldots, n-1\}$, so that for $\sigma \in 2^{<\omega}$, $|\sigma|$ is both the length and the domain of σ. For $\Sigma \subseteq 2^{<\omega}$, we write $[\Sigma]$ for the set $\{A \in 2^\omega \mid \forall n \in \mathbb{N}(A \restriction n \in \Sigma)\}$. We will also write, with some abuse of notation, $[\sigma] \cap 2^{<\omega}$ for the set $\{\tau \in 2^{<\omega} \mid \sigma \preceq \tau\}$ and $[\sigma] \cap 2^k$ for the set $\{\tau \in 2^k \mid \sigma \preceq \tau\}$.

A *Martin-Löf test* is a uniformly computably enumerable sequence $\langle \Sigma_n \subseteq 2^{<\omega} \mid n \in \mathbb{N} \rangle$ such that $\mu([\Sigma_n]) \leq 2^{-n}$, where μ is the Lebesgue measure. A Martin-Löf test *succeeds on* a real $A \in 2^\omega$ iff $A \in \bigcap_{n \in \mathbb{N}} [\Sigma_n]$. The set of reals on which a given Martin-Löf test succeeds is a null set; a Martin-Löf test is a particular notion of an effective null set. A real $A \in 2^\omega$ is *Martin-Löf random* iff no Martin-Löf test succeeds on it.

The notion of a Martin-Löf random can also be explained using martingales. We will define our martingales in different ways. What we give in the definitions will, however, always be enough so that given a real $A \in 2^\omega$ we can compute a function $d^A : \mathbb{N} \to \mathbb{R}_0^+$. Here $d^A(n)$ gives the player's capital after n bets. The martingale formulation of Martin-Löf randomness will show a closer connection to the other notions of randomness we will define and use.

We start by describing a *martingale* as a function $f : 2^{<\omega} \to \mathbb{R}_0^+$ (where $\mathbb{R}_0^+$ is the set of nonnegative reals) such that for all $\sigma \in 2^{<\omega}$ we have $f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}$, where σi denotes the concatenation of σ with the sequence $\langle i \rangle$. A martingale *succeeds on* a real $A \in 2^\omega$ iff $\limsup_{n \to \infty} f(A \restriction n) = \infty$.

To give an obviously equivalent definition more in line with the definitions given later, we can define the *capital function* $d^A : \mathbb{N} \to \mathbb{R}_0^+$ by setting $d^A(n) = f(A \restriction n)$, the capital after n bets on A. Then the martingale *succeeds* iff $\limsup_{n \to \infty} d^A(n) = \infty$ (this in particular means that $d^A(n) \downarrow$ for all n).

A martingale g is *effective* if there exists a computable function $\hat{g} : \mathbb{N} \times 2^{<\omega} \to \mathbb{Q}_0^+$ (where $\mathbb{Q}_0^+$ is the set of nonnegative rationals) which is nondecreasing in the first coordinate and such that $\lim_{n \to \infty} \hat{g}(n, \sigma) = g(\sigma)$.

Theorem 1 (Schnorr [Sch71Sch71, Sch71]). *A real is Martin-Löf random iff no effective martingale succeeds on it.*

A martingale is *(partial) computable* iff it is a (partial) computable function $f : 2^{<\omega} \to \mathbb{Q}_0^+$. A real is *(partial) computably random* iff no (partial) computable martingale succeeds on it. Clearly, any Martin-Löf random is partial computably random, and any partial computably random is computably random. However, these three notions do not coincide:

Theorem 2 (Ambos-Spies [AS98AS98, AS98]). *There are reals which are computably random but not partial computably random.*

Theorem 3 (Muchnik [MSU98MSU98, MSU98], Schnorr [Sch73Sch73, Sch73]). *There are reals which are partial computably random but not Martin-Löf random.*

Theorem 3 is a combination of Theorem 9.5 in Muchnik [MSU98MSU98, MSU98] and Theorem 3 in Schnorr [Sch73Sch73, Sch73]. The proofs of Theorems 1, 2, and 3 can also be found in [DHDH, DH] or [NieNie, Nie].

All the notions of randomness above have in common that the martingale bets on all the bits of the real in order, i.e., they bet monotonically. A nonmonotonic betting strategy has the flexibility to choose

which bits of the real to bet on (for instance, it might first bet on bit number 5, and then depending on the outcome bet on bit number 2 or 7, respectively). The exact definition (as taken essentially from $[\text{MMN}^+06\text{MMN}^+06, \text{MMN}^+06]$) is as follows.

A *scan rule* is a partial function $s : (\omega \times 2)^{<\omega} \to \omega$ such that for all $w \in (\omega \times 2)^{<\omega}$ we have that $s(w) \notin \text{dom}(w)$. Given the history of play—a sequence of bit locations and their values—the scan rule selects the next bit to bet on. The requirement $s(s) \notin \text{dom}(w)$ corresponds to not being allowed to bet on a bit you have already seen.

Given a real $A \in 2^{\omega}$, the scan rule selects a real $\tilde{A}$; however, the scan rule, and a betting strategy using it, use the full history of the play. This means that the object of interest is $\bar{A} : \omega \to (\omega \times 2)^{<\omega}$ defined as follows:

$$\bar{A}(0) = \big(s(\emptyset), A(s(\emptyset))\big),$$
$$\bar{A}(n) = \big(s(\bar{A} \upharpoonright n), A(s(\bar{A} \upharpoonright n))\big).$$

From this, the real played, $\tilde{A}$, can be defined by $\tilde{A}(n) = \pi_1(\bar{A}(n))$. Also, if $\tau \in 2^{<\omega}$, then we can in the same way define $\bar{\tau} \upharpoonright m$ for each m such that for all $n < m$, $s(\bar{\tau} \upharpoonright n) \in |\tau|$.

A *stake function* is a partial function $q : (\omega \times 2)^{<\omega} \to [0, 2]$. The stake function gives the bet towards the next bit selected being 0. A *nonmonotonic betting strategy* is a triple (λ, s, q) where $\lambda \in \mathbb{R}^+$ is the initial capital, s is a scan rule, and q a stake function. Define the capital after play n recursively by

$$d^A_{(\lambda,s,q)}(0) = \lambda,$$

$$d^A_{(\lambda,s,q)}(n+1) = \begin{cases} d^A_{(\lambda,s,q)}(n)q(\bar{A} \upharpoonright n) & \text{, if } \tilde{A}(n) = 0; \\ d^A_{(\lambda,s,q)}(n)(2 - q(\bar{A} \upharpoonright n)) & \text{, if } \tilde{A}(n) = 1. \end{cases}$$

We also use $d^{\sigma}_{(\lambda,s,q)}(n)$ when $\sigma \in 2^{<\omega}$. This is defined as follows: $d^{\sigma}_{(\lambda,s,q)}(n) = d^A_{(\lambda,s,q)}(n)$ for $A \in [\sigma]$ if all the bits used are from σ, otherwise $d^{\sigma}_{(\lambda,s,q)}(n)$ is undefined.

We say the betting strategy (λ, s, q) *succeeds* on A iff

$$\limsup_{n \to \infty} d^A_{(\lambda,s,q)}(n) = \infty.$$

A real is *Kolmogorov-Loveland random* if no computable nonmonotonic betting strategy succeeds on it. (Here, we may assume without loss of generality that all reals involved in any computable betting strategy, that is λ and the outputs of q, are actually rational. This assumption makes the notion simpler.)

The following theorem about these notions is well known.

Theorem 4 (Muchnik [MSU98MSU98, MSU98]). *Every Martin-Löf random real is Kolmogorov-Loveland random.*

However, the following question is a major open question in the area of randomness. It was first posed in Muchnik et al. [MSU98MSU98; MSU98, Question 8.11] (the wording is different there; chaotic is the same as ML-random, and unpredictable is the same as KL-random). It is also the last remaining open question from Ambos-Spies and Kučera [ASK00ASK00; ASK00, Open Problem 2.9] (the wording is different there, too; nonmonotonic computable random is the same as Kolmogorov-Loveland random, and Σ_1^0-random is the same as Martin-Löf random).

Question 5. *Do the notions of Martin-Löf randomness and Kolmogorov-Loveland randomness coincide?*

In Miller and Nies [MN06MN06, MN06], some weakenings of Kolmogorov-Loveland randomness are suggested as a way of approaching this question. They define restrictions of nonmonotonic betting strategies by how the sequence of bits bet on is generated.

Let $h : \omega \to \omega$ be an injection. Then we can bet on bit $h(n)$ in the nth round of betting: a betting strategy that uses h in the selection of bits is a betting strategy (λ, s, q) with $s(\sigma) = h(|\sigma|)$ for all $\sigma \in 2^{<\omega}$. We will write (λ, h, q) for the betting strategy in case s is computed from h in this fashion. (Thus the selection of bits no longer depends on the values of the previous bits bet on.)

Miller and Nies then use this to define several notions of randomness (where q is always a partial computable stake function): A real is *permutation random* if no partial betting strategy succeeds where h is any partial computable permutation of ω; and *injective random* iff no partial betting strategy succeeds where h is any partial computable injection. Since a betting strategy using an h that is not total does not succeed on any real, these notions stay the same if we only require h to be partial. These notions however would change if we required the stake function to be total (in fact, total permutation random (where both the permutation and the stake function are total) is the same as computably random).

It is not hard to see that Kolmogorov-Loveland randomness implies injective randomness, which in turn implies permutation randomness. Miller and Nies now ask whether one can at least separate the latter two notions from Martin-Löf randomness. In this paper, we show that injective randomness can be separated from Martin-Löf randomness:

Theorem 6. *There is a real $A \in 2^\omega$ which is injective random but not Martin-Löf random.*

We call (λ, h, q) a *partial computable permutation (resp. injective) martingale* if $\lambda \in \mathbb{Q}$, h is a partial computable permutation (resp. injection), and q is partial computable. Since it will always be enough to ensure that no martingale with initial capital 1 succeeds we will only deal with such martingale and write (h, q) for $(1, h, q)$.

To introduce many of the ideas we will use in an easier context we will first prove the following weaker theorem. We think the three main ideas in that proof (expected martingale, monotonizing, and the method of dealing with partiality) are also of independent interest.

Theorem 7. *There is a real $A \in 2^\omega$ which is permutation random but not Martin-Löf random.*

Note that the idea of an expected martingale, which is basic to all our considerations, was already used by Buhrman et al. [BvMR$^+$00BvMR$^+$00, BvMR$^+$00].

2. The Proof for Permutation Randomness

We need to construct a real $A \in 2^\omega$ and a computable function $g : \mathbb{N} \times 2^{<\omega} \to [0, \infty)$ (where we write $g_s(\nu)$ for $g(s, \nu)$) which is nondecreasing in the first coordinate such that $g = \lim_{s \to \infty} g_s(\sigma)$ is a martingale which succeeds on A and such that no partial permutation martingale succeeds on A. In fact, in our construction, for every s, $\sigma \mapsto g_s(\sigma)$ will be a martingale.

2.1. **Giving the Strategies Money.** In our construction, we will have certain strategies active at different nodes. These strategies will perform certain computations and as a result need to make certain bets, winning (or not losing too much) money along a certain string.

To have every possible strategy be able to do so, we note that $1 = \Sigma_{i=1}^{\infty} (\frac{1}{2})^i$. The ith active strategy can use capital $(\frac{1}{2})^i$ from the root. This means that if it is active at a node σ, at σ it has a capital of $c = (\frac{1}{2})^i 2^{|\sigma|}$ to use there. It will have either some finite number l or infinitely many substrategies. In case there are finitely many substrategies, each can use capital $\frac{c}{l}$ at σ, or equivalently $\frac{1}{l}(\frac{1}{2})^i$ from the root. In case there are infinitely many substrategies, the jth one can use capital $c(\frac{1}{2})^j$ at σ, or equivalently $(\frac{1}{2})^{i+j}$ from the root. If one of the substrategies succeeds at some stage s it finds a node τ extending σ satisfying certain properties. It will then change $g_s(\cdot)$ to $g_{s+1}(\cdot)$ *using its capital c' from the root along τ* as follows: for all $\nu \notin [\tau] \cup \{\eta \in 2^{<\omega} \mid \eta \prec \tau\}$,

$g_{s+1}(\nu) := g_s(\nu)$; for all $\nu \prec \tau$, $g_{s+1}(\nu) := g_s(\nu) + c'2^{|\nu|}$; and for all $\nu \in [\tau]$, $g_{s+1}(\nu) := g_s(\nu) + c'2^{|\tau|}$.

Organizing the construction this way, we will ensure that the martingale f we construct has initial capital less than 1. It will not have capital exactly equal to 1 as then it would be closely approximable by a computable martingale, something we know cannot happen (the martingale we construct wins on a real on which no partial permutation martingale wins; therefore certainly no computable martingale wins on it).

2.2. Combining Martingales.
The collection of monotonic martingales has some easy but important closure properties. If f and g are monotonic martingales, then so are cf, for any $c \in \mathbb{R}^+$, and $f + g$.

If we have an enumeration of martingales $\langle f_i \mid i \in \mathbb{N} \rangle$ with initial capital less than or equal to 1 and we want to find a real $A \in 2^\omega$ such that none of the martingales f_i succeeds on A, we can go about this as follows: First we find a σ_0 on which f_0 does not gain too much, i.e., $f_0(\sigma_0) < 2$ (in fact we can make sure it makes no gain at all). On σ_0, the martingale f_1 might have gained a lot, therefore set $s_1 = \frac{2 - f_0(\sigma)}{2f_1(\sigma)}$ and notice that then $f_0(\sigma_0) + s_1 f_1(\sigma_0) < 2$. So we can find an extension σ_1 of σ_0 where the martingale $f_0 + s_1 f_1$ does not gain too much; i.e., still has capital less than 2. Note that there f_0 or f_1 could have increased, but not too much, since $f_0(\sigma_1) < 2$ and $f_1(\sigma_1) < 2s_1$. If we can iterate this construction, we have found A as required (namely, $A = \bigcup_{i \in \mathbb{N}} \sigma_i$).

The difficulty is that since the martingales we have to beat are not monotonic, we cannot add them in this way. The way to overcome this difficulty is shown in the following sections.

2.3. Monotonizing a Martingale.
If (λ, s, q) is a total nonmonotonic martingale, then we can define $d^{\text{expec}}_{(\lambda,s,q)} : 2^{<\omega} \to [0, \infty)$, the expected capital function. We will work with the case that s is obtained from an injection $h : \mathbb{N} \to \mathbb{N}$ (slightly more general than we need) and initial capital 1. Then we can define $d^{\text{expec}}_{(h,q)}$ as follows:

For $\sigma \in 2^{<\omega}$, let $n_\sigma, l_\sigma \in \mathbb{N}$ be such that

$$l_\sigma > |\sigma|,$$
$$\operatorname{ran}(h \restriction n_\sigma) \subseteq l_\sigma, \text{ and}$$
$$\operatorname{ran}(h) \cap \operatorname{dom}(\sigma) = \operatorname{ran}(h \restriction n_\sigma) \cap \operatorname{dom}(\sigma).$$

This means that after n_σ many bets, all bets on σ will have been placed, and to complete these bets, no bets beyond the l_σ^{th} bet are needed. Then

define

$$d_{(h,q)}^{\text{expec}}(\sigma) := \sum_{\substack{\tau \in 2^{l_\sigma} \\ \sigma \prec \tau}} d_{(h,q)}^{\tau}(n_\sigma) 2^{-(l_\sigma - |\sigma|)},$$

where d^{τ} was defined on page 4.

Lemma 8. $d_{(h,q)}^{\text{expec}}$ *is a well-defined monotonic martingale.*

Intuitively, this lemma is clear from the probabilistic interpretation, but we give here a combinatorial proof.

Proof. In the context of this proof, we drop the subscript (h, q) from the martingales.

Given $\sigma \in 2^{<\omega}$, we have to show

(1) that in the computation of d^{expec}, for any values n and l which satisfy the requirements, we compute the same value; and
(2) that for all σ,

$$d^{\text{expec}}(\sigma) = \frac{d^{\text{expec}}(\sigma 1) + d^{\text{expec}}(\sigma 0)}{2}.$$

For (1), first let $n, l, l' \in \mathbb{N}$ be such that both pairs (n, l) and (n, l') satisfy the requirements in the definition of d^{expec} and such that $l' > l$. Then

$$
\begin{aligned}
\sum_{\sigma \prec \tau' \in 2^{l'}} d^{\tau'}(n) 2^{-(l'-|\sigma|)} &= \sum_{\sigma \prec \tau \in 2^{l}} \left(\sum_{\tau \prec \tau' \in 2^{l'}} d^{\tau'}(n) 2^{-(l'-|\sigma|)} \right) \\
&= \sum_{\sigma \prec \tau \in 2^{l}} \left(\sum_{\tau \prec \tau' \in 2^{l'}} d^{\tau}(n) 2^{-(l'-|\sigma|)} \right) \\
&= \sum_{\sigma \prec \tau \in 2^{l}} \left(d^{\tau}(n) 2^{-(l'-|\sigma|)} \sum_{\tau \prec \tau' \in 2^{l'}} 1 \right) \\
&= \sum_{\sigma \prec \tau \in 2^{l}} d^{\tau}(n) 2^{-(l'-|\sigma|)} 2^{l'-l} \\
&= \sum_{\sigma \prec \tau \in 2^{l}} d^{\tau}(n) 2^{-(l-|\sigma|)}.
\end{aligned}
$$

Next, let $n, l \in \mathbb{N}$ be such that the pair (n, l) satisfies the requirement in the definition of d^{expec} and such that $h(n+1) = i < l$. If $\tau \in 2^{l}$, then let $\tilde{\tau} \in 2^{l}$ be such that $\tilde{\tau}(i) = 1 - \tau(i)$, and such that for all $j < l$ with

$j \neq i$, $\tilde{\tau}(j) = \tau(j)$. Then (remembering the definition of $\bar{\tau}$ on page 4)

$$\sum_{\sigma \prec \tau \in 2^l} d^\tau(n)2^{-(l-|\sigma|)} =$$

$$= \frac{1}{2} \sum_{\sigma \prec \tau \in 2^l} \left(q(\bar{\tau} \upharpoonright n) + (2 - q(\bar{\tau} \upharpoonright n))\right) d^\tau(n)2^{-(l-|\sigma|)}$$

$$= \frac{1}{2} \sum_{\sigma \prec \tau \in 2^l} \left((q(\bar{\tau} \upharpoonright n)d^\tau(n)2^{-(l-|\sigma|)}) + (2 - q(\bar{\tau} \upharpoonright n))d^\tau(n)2^{-(l-|\sigma|)}\right)$$

$$= \frac{1}{2} \sum_{\sigma \prec \tau \in 2^l} \left(d^\tau(n+1)2^{-(l-|\sigma|)} + d^{\tilde{\tau}}(n+1)2^{-(l-|\sigma|)}\right)$$

$$= \frac{1}{2} \left(\sum_{\sigma \prec \tau \in 2^l} d^\tau(n)2^{-(l-|\sigma|)} + \sum_{\sigma \prec \tau \in 2^l} d^{\tilde{\tau}}(n)2^{-(l-|\sigma|)}\right)$$

$$= \sum_{\sigma \prec \tau \in 2^l} d^\tau(n+1)2^{-(l-|\sigma|)}.$$

Note that in the third equality, the terms might be reordered (depending on whether $\tau(i) = 0$).

This shows that in the definition of d^{expec}, the exact values of n and l are irrelevant as long as they are big enough. It remains to show (2), i.e., that d^{expec} satisfies the martingale equation. So let both n and l be large enough, then

$$d^{\mathrm{expec}}(\sigma) = \sum_{\sigma \prec \tau \in 2^l} d^\tau(n)2^{-(l-|\sigma|)}$$

$$= \sum_{\substack{\sigma \prec \tau \in 2^l \\ \tau(|\sigma|)=0}} d^\tau(n)2^{-(l-|\sigma|)} + \sum_{\substack{\sigma \prec \tau \in 2^l \\ \tau(|\sigma|)=1}} d^\tau(n)2^{-(l-|\sigma|)}$$

$$= \frac{1}{2}d^{\mathrm{expec}}(\sigma 0) + \frac{1}{2}d^{\mathrm{expec}}(\sigma 1).$$

$\square$

There are now two problems to overcome. Firstly, we need to see that we can use d^{expec} to beat the original nonmonotonic martingale, and secondly, that we can have a sufficiently computable version of it.

The problem with seeing that d^{expec} succeeds on the same reals that the nonmonotonic martingale (s, q) succeeds on is simplified by taking the slowly-but-surely winning version of (s, q) (also known as the saving version of (s, q)). The problem solved by this is that d^{expec} does not reflect all fluctuations that appear in the capital history of (s, q).

Lemma 9. *Let (s, q) be a partial computable nonmonotonic martingale. Then there exists a partial computable nonmonotonic martingale $(\bar{s}, \bar{q})$ that succeeds on the same reals as (s, q) where the capital function $\bar{d}$ of $(\bar{s}, \bar{q})$ satisfies*

$$\forall \sigma \, \forall \tau \, (\bar{d}(\sigma\tau) > \bar{d}(\sigma) - 2), \ \text{and}$$
$$\forall \sigma \, (\bar{d}(\sigma) < 2(|\sigma| + 1)).$$

A version of this well known lemma can be found as [BvMR$^+$00BvMR$^+$00; BvMR$^+$00, Lemma 2.3, p. 579], and the computations in their proof immediately generalize to this context, so we will not repeat them here. The intuition is that in the betting of the martingale every time your capital increases to more than 2, you take 2 from your capital and "keep it in the bank" and only continue betting with the remaining little bit of capital. This way your capital can never decrease by more than 2. If the original martingale succeeds, then infinitely often the little bit of capital you are betting with will increase above 2 so that this martingale succeeds as well.

Now for a nonmonotonic martingale (s, q), define $d^{\text{ss}-\text{expec}}$ to be the expected capital function computed for the slowly-but-surely winning version $(\bar{s}, \bar{q})$ of (s, q).

Lemma 10. $d^{\text{ss}-\text{expec}}$ *is a monotonic martingale which succeeds every real on which (s, q) succeeds.*

Proof. Let $A \in 2^\omega$ be a real on which (s, q) succeeds. Then $(\bar{s}, \bar{q})$ also succeeds on A. We need to see that for every L, there is an $n \in \mathbb{N}$ such that $d^{\text{ss}-\text{expec}}(n)$ as computed along A is greater than L.

Let k be such that $\bar{d}(k) > L + 2$, and let $\sigma \prec A$ be such that all bets that are made in the computation of $\bar{d}(k)$ are made on σ. This means that by the slowly-but-surely winning condition for all τ extending σ and all $l \geq k$, we have $\bar{d}^\tau(l, \tau) > L$. This in turn implies that $d^{\text{ss}-\text{expec}}(\sigma) > L$ as was to be proven. $\square$

The final bit of analysis of $d^{\text{ss}-\text{expec}}$ that is needed is to come up with a usable condition under which we can compute it. Existence of n_σ and l_σ for any σ is clear, but it is not clear under what conditions they can be found computably. We next give such conditions.

Say we have already decided on σ_0 as the initial segment of the real A we are constructing, and that we now want to extend it to length $k > |\sigma_0|$. Then we want to be able to compute $d^{\text{ss}-\text{expec}}$ on $[\sigma_0] \cap 2^k$.

Lemma 11. *For a total nonmonotonic martingale (s, q) where q is obtained from an injective $h : \mathbb{N} \to \mathbb{N}$ and σ_0 and k are as in the paragraph above, we can compute $d^{\mathrm{ss-expec}}$ on $[\sigma_0] \cap 2^k$ from $|\operatorname{ran}(h) \cap k|$.*

Proof. From $|\operatorname{ran}(h) \cap k|$, we can compute an n such that $\operatorname{ran}(h \upharpoonright n) \cap k = \operatorname{ran}(h) \cap k$, i.e., after n many bets, all bets on every τ of length k will have been made. This means that $d^{\mathrm{ss-expec}}$ can be computed on any τ using $n_\tau = n$ and $l_\tau = \max\{\operatorname{ran}(h \upharpoonright n)\}$. $\qquad\qquad\square$

Note that the hypothesis needed for $d^{\mathrm{ss-expec}}$ to be computable is obviously satisfied for permutation martingales as then always $|\operatorname{ran}(h) \cap k| = k$.

2.4. Partiality. Initially, dealing with partial permutation martingales seems straightforward. Once you have decided on σ_0 as an initial segment for the real and (h, q) is the next permutation martingale to consider, the strategy working under the assumption that (h, q) is partial should just look for an extension τ of σ_0 where (h, q) diverges. The problem with this is that the earlier martingales already considered, of which all the ones which are "sufficiently total" have been combined into a single monotonic martingale f_{mon}, might make a large gain on each such τ.

The situation we have is $\sigma \in 2^{<\omega}$, a monotonic martingale f_{mon} which is total on $[\sigma] \cap 2^{<\omega}$ and for which $f_{\mathsf{mon}}(\sigma) < 2$, and a partial permutation martingale (h, q). We have to find $\tau \succ \sigma$ such that either $d_{(h,q)}$ diverges on τ, or we have a method of adding $d_{(h,q)}$ to f_{mon}.

We need to be explicit about what we mean when $d_{(h,q)}$ diverges on τ. For this, we define $d^\tau_{(h,q)} \uparrow^*$ by

$$\exists i \left(\operatorname{ran}(h \upharpoonright i) \subsetneq |\tau| \wedge h(i) \uparrow \right) \vee \exists i \left(\operatorname{ran}(h \upharpoonright i) \subseteq |\tau| \wedge \exists n < i \left(q(\bar{\tau} \upharpoonright n) \uparrow \right) \right),$$

and $d^\tau_{(h,q)} \downarrow^*$ as its negation (note that it will often be the case that for $\tau \prec \tau'$ we have that $d^\tau_{(h,q)} \downarrow^*$ and $d^{\tau'}_{(h,q)} \uparrow^*$; this happens for instance when $h(0) \in |\tau'| \setminus |\tau|$ and $h(1)$ diverges).

The case distinction which needs to be made is the following:
Either

(C:↑) $\qquad\qquad \exists \tau \succ \sigma \left(f_{\mathsf{mon}}(\tau) < 2 \wedge d^\tau_{(h,q)} \uparrow^* \right),$

or

(C:↓) $\qquad\qquad \forall \tau \succ \sigma \left(f_{\mathsf{mon}}(\tau) < 2 \to d^\tau_{(h,q)} \downarrow^* \right).$

The strategy is then as follows: In case (C:↑), we search for such a τ. That is, at stage s, we assume that any computation which

does not converge within s steps diverges, and we look for the length-lexicographically first τ which satisfies (C:$\uparrow$).

In case (C:$\downarrow$), we will find a total (on $[\sigma]$) permutation martingale $\tilde{d}_{(h,q)}$ which is equal to $d_{(h,q)}$ everywhere where f_{mon} is less than 2. The idea is that in order to compute $\tilde{d}^{\tau}_{(h,q)}(n)$, we find the largest part of τ where f_{mon} is less than 2, and give $\tilde{d}$ on τ the value of $d_{(h,q)}$ on that part. Essentially this means as soon as we lose the guarantee — from the assumption that we are in case (C:$\downarrow$) — that $d_{(h,q)}$ converges, we always bet even.

We need to be somewhat careful with what "largest part" means, since we do need the property that if ν_0 and ν_1 are such that $\nu_0 \upharpoonright \mathrm{ran}(h \upharpoonright j) = \nu_1 \upharpoonright \mathrm{ran}(h \upharpoonright j)$ (that is ν_0 and ν_1 agree on the bit values of the first j places inspected) then for all $i < j$, $\tilde{d}^{\nu_0}_{(h,q)}(i) = \tilde{d}^{\nu_1}_{(h,q)}(i)$.

The martingale is now defined as follows: To compute $\tilde{d}^{\tau}_{(h,q)}(n)$ for $\tau \succ \sigma$, search for the maximum $j \leq n$ such that there is a $\tau' \succ \sigma$ satisfying $\tau' \upharpoonright \mathrm{ran}(h \upharpoonright j) = \tau \upharpoonright \mathrm{ran}(h \upharpoonright j)$ and $f_{\mathsf{mon}}(\tau') < 2$. Then set $\tilde{d}^{\tau}_{(h,q)}(n) := d^{\tau'}_{(h,q)}(j)$. Note that this converges by the case assumption (and is equal to $d^{\tau}_{(h,q)}(j)$).

2.5. Putting it all together.
Let $\langle (h_i, q_i) \mid i \in \mathbb{N} \rangle$ be an enumeration of all partial permutation computable martingales with initial capital 1. We need to find an $A \in 2^{\omega}$ on which none of these martingales succeeds (showing that A is permutation random) and construct an effective martingale g which does succeed on A (showing that A is not Martin-Löf random). We will define $\langle g_i \mid i \in \mathbb{N} \rangle$ so that $g(\sigma) = \lim_{i \to \infty} g_i(\sigma)$ and the sequence of g_i is uniformly computable. Let $g_0 : 2^{<\omega} \to \mathbb{R}^+_0$ be the constant 0 martingale.

The strategy $N(i, \sigma, \epsilon_0, \ldots, \epsilon_{i-1})$ (where, from now on, we will write $\bar{\epsilon}$ for $\epsilon_0, \ldots, \epsilon_{i-1}$) is the strategy with the following parameters

- i, the number of martingales supposedly already dealt with on σ,
- σ, indicating the cone, $[\sigma]$, on which this strategy will act, and
- $\epsilon_j \in \{0, 1\}$ (for $j < i$), denoting whether the previous strategies were able to find a τ where (h_j, q_j) diverges and $f_j^{\bar{\epsilon} \upharpoonright j+1}(\tau) < 2$ (if $\epsilon_j = 0$) or where (h_j, q_j) converges on every extension τ of σ where $f_j^{\bar{\epsilon} \upharpoonright j+1}(\tau) < 2$ (if $\epsilon_j = 1$). Here and below, $f_j^{\bar{\epsilon} \upharpoonright j+1}$ is one of the monotonic martingales constructed by the strategy $N(j, \sigma', \bar{\epsilon} \upharpoonright j)$ where $\sigma' \prec \sigma$.

This strategy performs two kinds of actions:

- it will start new strategies $N(i + 1, \tau, \bar{\epsilon}\epsilon)$ where $\sigma \prec \tau$ and $\epsilon \in \{0, 1\}$, and
- it will at various times define g_{s+1} from g_s.
- it will (attempt to) define $f_i^{\bar{\epsilon}\epsilon}$ for $\epsilon \in \{0, 1\}$.

In fact, whenever a new strategy is started a new g_{s+1} will also be defined. It is important to note here that s is not determined by this strategy, but by the overall results of all strategies active so far. In all cases, s will be the maximum value such that g_s is defined. If we think of defining g_{s+1} from g_s as modifying the martingale g, then s is the number of times we have already modified g before (this is the terminology we will use below).

If $N(i, \sigma, \bar{\epsilon})$ is the j-th active strategy, then the strategy as a whole can use capital $C := (\frac{1}{2})^j$ from the root (as explained in Section 2.1). Since this strategy will possibly infinitely often modify g, it partitions $C = \sum_{l \in \mathbb{N}} c_l$ (where all $c_l > 0$) and uses these parts from C as appropriate (c_0 is used for the case (C:$\downarrow$), and c_{l+1} is used for the l-th time the actions for the case (C:$\uparrow$) modify g).

Define $\hat{f}_j^{\bar{\epsilon}|j}$ to be the monotonic martingale $\tilde{d}_{(h_j, q_j)}^{\text{ss-expec}}$ determined from $d_{(h_j, q_j)}$ and the monotonic martingale $f_{j-1}^{\bar{\epsilon}|j}$ as in the previous section (where $f_{-1}^{\emptyset} \equiv 0$).

In case $\forall \nu \succ \sigma \, (f_{i-1}^{\bar{\epsilon}}(\nu) < 2 \rightarrow d_{(h_i, q_i)}^{\nu} \downarrow^*)$, i.e., in case (C:$\downarrow$), $N(i, \sigma, \bar{\epsilon})$ has to find a long enough extension τ of σ such that $f_i^{\bar{\epsilon}1}(\tau) := f_{i-1}^{\bar{\epsilon}}(\tau) + s_i \hat{f}_i^{\bar{\epsilon}}(\tau) < 2$, where s_i is determined below in the second substrategy. In the other case, (C:$\uparrow$), it has to find a long enough extension of a τ such that $f_i^{\bar{\epsilon}0}(\tau) := f_{i-1}^{\bar{\epsilon}}(\tau) < 2$ and $d_{(h_i, q_i)}^{\tau} \uparrow^*$, which then exists. The next paragraph explains what long enough means.

Since we need to ensure g wins on the real A we construct, and the τ are approximations to this A, the strategy $N(i, \sigma, \bar{\epsilon})$ ensures that $g(\tau) \geq 2^i$. So for case (C:$\downarrow$) it looks for a τ such that $c_0 2^{|\tau|} \geq 2^i$, and in case (C:$\uparrow$) if active for the l-th time we look for a τ such that $c_{l+1} 2^{|\tau|} \geq 2^i$.

At stage s in the construction, this strategy computes everything for at most s steps.

The <u>first substrategy</u> looks for the length-lexicographically least $\tau' \succ \sigma$ such that $d_{(h_i, q_i)}^{\tau'} \uparrow^*$ and $f_{i-1}^{\bar{\epsilon}}(\tau') < 2$, and then for the least $\tau \succ \tau'$ such that $f_{i-1}^{\bar{\epsilon}}(\tau) < 2$ and $c_{l+1} 2^{|\tau|} > 2^i$ where l is the number of times this substrategy has been active before. If this τ' is different from the τ' we found in earlier stages (which means that the computation on that earlier τ' has converged), this substrategy becomes active. We then stop the previous strategies $N(i + 1, \tau'', \bar{\epsilon}0)$ we started, and start

$N(i+1, \tau, \bar{\epsilon}0)$. Also, we define g_{s+1} from g_s using capital c_{l+1} from the root along τ' (as explained in Section 2.1), where s is the number of times we have already modified the martingale g before.

It is clear that in case (C:↑) (where $\sigma = \sigma$, $f_m = f^{\bar{\epsilon}}_{i-1}$, and $(h, q) = (h_i, q_i)$) and where $\bar{\epsilon}$ is correct, this strategy succeeds; it will find a pair (τ, τ') which permanently satisfies the requirement.

Simultaneously, for the <u>second substrategy</u>, we wait for a stage at which $\hat{f}^{\bar{\epsilon}}_i$ and $f^{\bar{\epsilon}}_{i-1}$ converge on σ. Then we set $s_i := \frac{1}{2} \frac{2 - f^{\bar{\epsilon}}_{i-1}(\sigma)}{\hat{f}^{\bar{\epsilon}}_i(\sigma)}$ (to ensure $f^{\bar{\epsilon}1}_i(\sigma) < 2$). After this stage, we search for the length-lexicographically least $\tau \succ \sigma$ such that $f^{\bar{\epsilon}1}_i(\tau) < 2$ and $c_0 2^{|\tau|} > 2^i$. If we find such a τ, we start the strategy $N(i+1, \tau, \bar{\epsilon}1)$ and define g_{s+1} from g_s using capital c_0 from the root along τ (as explained in Section 2.1), where s is the number of times we have already modified the martingale g before.

It is clear that in case (C:↓) (where $\sigma = \sigma$, $f_m = f^{\bar{\epsilon}}_{i-1}$, and $(h, q) = (h_i, q_i)$) and where $\bar{\epsilon}$ is correct, this strategy succeeds.

We start the construction by starting $N(0, \emptyset, \emptyset)$.

2.6. **Verification.** Recursively (but certainly not computably!), define σ_i and ϵ_i (where $\sigma_{-1} = \emptyset$) as follows.

We set $\epsilon_i = 1$ if (C:↓) is true for $\sigma = \sigma_{i-1}$, $f_{\mathsf{mon}} = f^{\bar{\epsilon}}_{i-1}$, and $(h, q) = (h_i, q_i)$. In that case, we set $\sigma_i = \tau$ where τ is found by the strategy $N(i, \sigma_{i-1}, \bar{\epsilon} \upharpoonright i)$ by its second substrategy. Note that then $N(i+1, \sigma_i, \bar{\epsilon} \upharpoonright i, 1)$ is started.

We set $\epsilon_i = 0$ if (C:↑) is true for $\sigma = \sigma_{i-1}$, $f_{\mathsf{mon}} = f^{\bar{\epsilon}}_{i-1}$, and $(h, q) = (h_i, q_i)$. In that case, we let $\tau' \in 2^{<\omega}$ be the length-lexicographically least element of $[\sigma_{i-1}] \cap 2^{<\omega}$ for which $d^{\tau'}_{(h_i, q_i)} \uparrow^*$, $f^{\bar{\epsilon}}_{i-1}(\tau') < 2$, and s is a stage at which for all ν length-lexicographically before τ', $d^{\nu}_{(h_i, q_i)}$ converges in fewer than s steps if $f^{\bar{\epsilon}}_{i-1}(\nu) < 2$. Then, at stage s, the first substrategy of $N(i, \sigma_{i-1}, \bar{\epsilon} \upharpoonright i)$ will pick τ' as well as an extension τ of τ' (if it hadn't already done so at an earlier stage), and we will never again find a new pair (τ', τ). Then set $\sigma_i = \tau$. Note that then $N(i+1, \sigma_i, \bar{\epsilon} \upharpoonright i, 0)$ is started and never stopped thereafter.

Define $A := \bigcup_{i \in \mathbb{N}} \sigma_i$. Note that clearly g succeeds on A since $g(\sigma_i) \geq 2^i$ (as a result of the modification of g done by $N(i, \sigma_{i-1}, \bar{\epsilon} \upharpoonright i)$ when

$N(i+1, \sigma_i, \bar{\epsilon} \upharpoonright i+1)$ was started). Note that A is Δ_3^0, since

$$\sigma \prec A \Leftrightarrow \exists \bar{\epsilon}(\bar{\epsilon} \text{ is correct} \wedge \sigma \text{ is obtained when running}$$
$$\text{the construction above knowing } \bar{\epsilon})$$
$$\Leftrightarrow \forall \bar{\epsilon}(\bar{\epsilon} \text{ is correct} \rightarrow \sigma \text{ is obtained when running}$$
$$\text{the construction above knowing } \bar{\epsilon}).$$

Here the statement "$\bar{\epsilon}$ is correct" is Δ_3^0, and with that information determining the outcome of the construction is Σ_2^0.

We need to see that none of the partial computable permutation martingales (h_i, q_i) succeeds on A. Suppose that (h_i, q_i) succeeds on A. We will derive a contradiction to the fact that for all $i \in \mathbb{N}$, $f_i^{\bar{\epsilon} \upharpoonright i+1}(\sigma_i) < 2$.

We know that if (h_i, q_i) succeeds on A, then $\hat{\tilde{d}}_i^{\bar{\epsilon} \upharpoonright i}$ also succeeds on A (note that this in particular implies that (C:↓) is the correct case for strategy $N(i, \sigma_{i-1}, \bar{\epsilon} \upharpoonright i)$, i.e., $\epsilon_i = 1$). Pick γ such that $\sigma_i \prec \gamma \prec A$ and $\hat{\tilde{d}}_i^{\bar{\epsilon} \upharpoonright i}(\gamma) > \frac{2}{s_i} + 2$. Since $\hat{\tilde{d}}_i^{\bar{\epsilon} \upharpoonright i}$ is a slow-but-sure martingale, this means that for all $\nu \succ \gamma$, we have $\hat{\tilde{d}}_i^{\bar{\epsilon} \upharpoonright i}(\nu) > \frac{2}{s_i}$. This holds in particular for any j such that $\sigma_j \succ \gamma$ (which implies $j \geq i$). But this in turn implies that

$$f_j^{\bar{\epsilon} \upharpoonright j+1}(\sigma_j) = \sum_{\substack{l \leq j \\ \epsilon_l = 1}} s_l \hat{\tilde{d}}_l^{\bar{\epsilon} \upharpoonright l}(\sigma_j) > s_i \hat{\tilde{d}}_i^{\bar{\epsilon} \upharpoonright i}(\sigma_j) > 2,$$

which contradicts the choice of σ_j, showing that (h_i, q_i) does not succeed on A.

3. The Proof for Injective Randomness

We again need to construct a real $A \in 2^\omega$ and a computable function $g : \mathbb{N} \times 2^{<\omega} \to [0, \infty)$ which is nondecreasing in the first coordinate and such that $g = \lim_{s \to \infty} g_s$ is a martingale which succeeds on A. This time no partial injective martingale can succeed on A.

When dealing with partial injective martingales we have no hope of producing a martingale with the same good properties the expected martingale had for partial permutation martingales. We were able to come up with a sufficient approximation, however. The idea is to ignore most bits, and to only look for bits of a given type. If this type is chosen, or rather guessed, correctly then we can use this approximation, the average operator, to construct a sequence of clopen sets of decreasing measure (a Martin-Löf test). We can then prove that in the intersection of these clopen sets there is a real on which none of the partial injective martingales wins. Since we do not know the correct

guesses, we construct a Martin-Löf martingale that combines all our
different attempts, and in particular will include all the correct guesses.

Let $\langle f_i = (h_i, q_i) \mid i \in \mathbb{N} \rangle$ be an enumeration of all partial injective
martingales (with initial capital 1); without loss of generality we can
assume that all these martingales are slow-but-sure.

3.1. **Types of Bits.** The main difficulty to solve in this proof is that
not all martingales bet on all bits. This means that during the con-
struction you never know whether the betting on a bit is done (un-
less you happen to stumble upon a bit on which all martingales you
are considering make a bet). The solution is to make guesses as to
which martingales will bet on a bit. This does not work completely
straightforward—we need a method to bring the number of guesses we
have to make down to a manageable number.

Let us assume we are in the part of the construction where we are
considering the first $n+1$ martingales, $f_0, \ldots, f_n$. Let $A \in {}^{\mathbb{N}}2$ be a real
and $k \in \mathbb{N}$ a bit location. If $T \subseteq \{0, \ldots, n\}$ is such that during the
betting, the martingales f_i bet on location k iff $i \in T$, then we know
when the betting on location k is done (that is when all f_i for $i \in T$
have bet on bit k). Not knowing what the appropriate T is, we need to
guess for it. If our guess T is correct for infinitely many bit locations
then we could hope to use it in our construction.

However, the locations for which T is correct cannot be recognized
during the construction, since there might also be infinitely many bits
for which the correct guess is a proper superset of T. If then we see all
martingales in T bet on a location and we act on this, we might still
act inappropriately since more martingales might bet on this location.
The solution is to not just guess for such T but to guess for maximal
such T. We work this out in detail below.

Definition 12. We say martingale $f_i = (h_i, q_i)$ bets on bit k iff there
is an $n \in \mathbb{N}$ such that $h_i(n) = k$.

Note that since the locations the martingale f_i bets on are deter-
mined by the injection h_i, they do not depend on the real, i.e., if f_i
bets along $A \in 2^\omega$ bets on location k, then for all $B \in 2^\omega$ f_i, bets
on location k when betting on B. This shows that the notion in the
next definition is well-defined. Also note that even when f_i bets on a
bit k, it might still be the case that the partial function q_i does not
sufficiently converge along a certain real to compute the value of the
martingale.

Definition 13. The set $\mathsf{type}_n(k) = \mathsf{type}(n,k) \subseteq \{0,\ldots,n\}$ is defined by $i \in \mathsf{type}_n(k)$ iff $i \leq n$ and f_i bets on bit k. We will say k is *of n-type T* iff $\mathsf{type}_n(k) = T$.

We need the following observations:

- For all $n \in \mathbb{N}$,

$$\mathbb{N} = \coprod_{T \subseteq \{0,\ldots,n\}} \{k \in \mathbb{N} \mid \mathsf{type}_n(k) = T\}$$

  (here $\coprod$ denotes disjoint union).
- If k is of n-type T, then when all f_i for $i \in T$ have bet on k no more bets on k will be made. More precisely, if t is such that for all $i \in T$ we have $k \in \mathrm{ran}(h_i \restriction (t+1))$ then for all $t' > t$ and all $i \leq n$ we have that $h_i(t') \neq k$.
- Let $K \in \mathbb{N}$. <u>If</u> T is $\subseteq$-maximal in

$$\{T \subseteq \{0,\ldots,n\} \mid \exists k > K \; \mathsf{type}(n,k) = T\},$$

  and we find $k > K$ and t such that for all $j \in T$, martingale f_j bets on k before or on bet t ($k \in \mathrm{ran}(h_i \restriction (t+1))$), <u>then</u> $\mathsf{type}(n,k) = T$ and no more bets will be made on k.
- If T is $\subseteq$-maximal in

$$\{T \subseteq \{0,\ldots,n\} \mid \exists^\infty k > l \; \mathsf{type}(n,k) = T\},$$

  then there is a $K \in \mathbb{N}$ such that s is $\subseteq$-maximal in

$$\{T \subseteq \{0,\ldots,n\} \mid \exists k > K \; \mathsf{type}(n,k) = T\}.$$

The last observation motivates the guesses we will make; our guesses will be of the form (K,T). And this represents the guess that T is a maximal type appearing infinitely often, and K is big enough so that the finitely many bits whose type is a proper superset of T appear below level K.

3.2. **The Average Operator.** In this subsection, we work under the assumption that the martingales involved are total. In the next subsection, we show how to deal with partiality.

Let $\sigma : \omega \rightharpoonup 2$ be finite (i.e. σ is a finite partial map from ω to 2) and t a number such that for all $k \in \mathrm{dom}(\sigma)$ if $j \in \mathsf{type}(n,k)$, then f_j has bet on k before its $(t+1)^{\mathrm{st}}$ bet, that is t is so large that after t bets all bets that will be made on σ have been made. Now let l be such that all bets by martingales f_j ($j \leq n$) that are made before the $(t+1)^{\mathrm{st}}$ bet are made on bits $k < l$ ($\mathrm{ran}(h_j \restriction (t+1)) \subseteq \{0,\ldots,l-1\}$).

If (t,l) satisfies the requirements in the previous paragraph then we say (t,l) is *sufficiently out there (for σ and n)*.

For (t, l) sufficiently out there, define

$$\operatorname{Av}_n^{(t,l)}(\sigma) := \sum_{\substack{\tau \in 2^l \\ \sigma \prec \tau}} 2^{-(l-|\sigma|)} (\sum_{j \leq n} f_j^t(\tau)).$$

The following two lemmas show some essential properties of Av.

Lemma 14. *If both (t_0, l_0) and (t_1, l_1) are sufficiently out there for σ and n, then $\operatorname{Av}_n^{(t_0,l_0)}(\sigma) = \operatorname{Av}_n^{(t_1,l_1)}(\sigma)$.*

This follows immediately from the martingale property of the f_j $(j \leq n)$. It shows that we can write $\operatorname{Av}(\sigma)$ for $\operatorname{Av}^{(t,l)}(\sigma)$ where (t, l) is any pair that is sufficiently out there.

Lemma 15. *If $k \notin \operatorname{dom}(\sigma)$, $\sigma_0 = \sigma \cup \{(k, 0)\}$, and $\sigma_1 = \sigma \cup \{(k, 1)\}$, then*

$$\operatorname{Av}_n(\sigma) = \frac{\operatorname{Av}_n(\sigma_0) + \operatorname{Av}_n(\sigma_1)}{2}.$$

This is proved exactly as Lemma 8 after choosing (t, l) sufficiently out there for all three computations.

Definition 16. *If $T \subseteq \{0, \ldots, n\}$ we define the restricted Av operator, Av_T, as follows:* Let (t, l) be sufficiently out there for n and σ. Then

$$\operatorname{Av}_T(\sigma) = \sum_{\substack{\tau \in 2^l \\ \sigma \prec \tau}} 2^{-(l-|\sigma|)} (\sum_{j \in T} f_j^t(\tau)).$$

Clearly the analogues of Lemma 14 and Lemma 15 hold for Av_T. Note also that there is a weaker notion of (t, l) being *T-sufficiently out there* for σ which just requires (t, l) to be such that $\operatorname{dom}(\sigma) \cap \operatorname{ran}(h_i \restriction t) = \operatorname{dom}(\sigma) \cap \operatorname{ran}(h_i)$ for $i \in T$ and $\max(\operatorname{ran}(h_i \restriction t)) \leq l$. Then just like in Lemma 14 the exact value of (t, l) does not influence the value of Av_T computed using it.

Lemma 17. *Let $k \notin \operatorname{dom}(\sigma)$, let bit k be of type T, and $T \subseteq S \subseteq \{0, \ldots, n\}$. If $\operatorname{Av}_T(\sigma \cup \{(k, i)\}) \leq \operatorname{Av}_T(\sigma \cup \{(k, 1-i)\})$, then $\operatorname{Av}_S(\sigma \cup \{(k, i)\}) \leq \operatorname{Av}_S(\sigma \cup \{(k, 1-i)\})$.*

We see this is true by the following computation, where σ_i denotes $\sigma \cup \{(k, i)\}$ and (t, l) is sufficiently out there:

$$
\begin{aligned}
\mathrm{Av}_S(\sigma_i) &= \sum_{\substack{\tau \in 2^l \\ \sigma_i \prec \tau}} 2^{-(l-|\sigma_i|)} \Big(\sum_{j \in S} f_j^t(\tau) \Big) \\
&= \sum_{\substack{\tau \in 2^l \\ \sigma_i \prec \tau}} 2^{-(l-|\sigma_i|)} \Big(\sum_{j \in S \setminus T} f_j^t(\tau) \Big) + \sum_{\substack{\tau \in 2^l \\ \sigma_i \prec \tau}} 2^{-(l-|\sigma_i|)} \Big(\sum_{j \in T} f_j^t(\tau) \Big) \\
&\overset{*}{=} \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} \Big(\sum_{j \in S \setminus T} f_j^t(\tau) \Big) + \mathrm{Av}_T(\sigma_i) \\
&\leq \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} \Big(\sum_{j \in S \setminus T} f_j^t(\tau) \Big) + \mathrm{Av}_T(\sigma_{1-i}) \\
&= \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} \Big(\sum_{j \in S \setminus T} f_j^t(\tau) \Big) + \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} \Big(\sum_{j \in T} f_j^t(\tau) \Big) \\
&= \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} \Big(\sum_{j \in S} f_j^t(\tau) \Big) \\
&= \mathrm{Av}_S(\sigma_{1-i}).
\end{aligned}
$$

Equality $(*)$ follows since $|\sigma_i| = |\sigma_{1-i}|$ and for $j \in S \setminus T$ the martingale f_j does not bet on bit k.

What this all achieves is that when we have a correct guess for the type, we can computably find which of zero or one does not increase the average value.

3.3. **Partiality.** We are going to use a similar strategy to the case of partial permutation martingales to deal with partiality. Note that we cannot define monotone martingales associated to partial injective martingales, so the details will look different.

Let σ denote the partial string (partial function $\omega \rightharpoonup \{0, 1\}$) that has already been determined, P a set of indices for which earlier strategies have determined that the associated martingales are partial, $n \in \mathbb{N}$ the index of the next martingale to consider, and for $j \in \{0, \ldots, n-1\} \setminus P$, write $\tilde{f}_j$ for the total version of $f_j = (h_j, q_j)$. We write $\widetilde{\mathrm{Av}}$ for Av computed using the $\tilde{f}_j$. The two cases to consider are then the following.

(CI:↑) $\qquad \exists \tau \succ \sigma \left(\widetilde{\mathrm{Av}}_{(\{0,\ldots,n-1\} \setminus P)}(\tau) < 2 \wedge d_{(h_n, q_n)}^\tau \uparrow^* \right),$

or

$$(\text{CI:}\downarrow) \qquad \forall \tau \succ \sigma \left(\widetilde{\text{Av}}_{(\{0,\ldots,n-1\}\setminus P)}(\tau) < 2 \rightarrow d^{\tau}_{(h_n,q_n)} \downarrow^* \right).$$

The strategy is then as follows: In case (CI:↑), we search for such a τ. That is, at stage s, we assume that any computation diverges if it does not converge within s steps, and we look for the length-lexicographically first τ which satisfies (CI:↑).

In case (CI:↓), we find a total permutation martingale $\tilde{f}_n$ which is equal to (h_n, q_n) everywhere where the average of the previous martingales is less than 2. This martingale is defined as follows: To compute $\tilde{f}_n = \tilde{d}^{\tau}_{(h,q)}(n)$ for $\tau \succ \sigma$, search for the maximum $m \leq n$ such that there is a $\tau' \succ \sigma$ satisfying $\tau' \restriction \text{ran}(h \restriction m) = \tau \restriction \text{ran}(h \restriction m)$ and $\widetilde{\text{Av}}_{(\{0,\ldots,n-1\}\setminus P)}(\tau') < 2$. Then set $\tilde{d}^{\tau}_{(h,q)}(n) := d^{\tau'}_{(h,q)}(m)$. Note that this converges by the case assumption (and is equal to $d^{\tau}_{(h,q)}(m)$).

3.4. The Strategy/Construction.

Here we describe the overall strategy, i.e., the construction of our Martin-Löf martingale. We use a similar idea as before, starting many different substrategies with associated capital. If they succeed at stage s, they construct g_{s+1} from g_s by adding a computable martingale d_C to g_s. Here C is a clopen set determined by the strategy, and $d_C(\sigma) = 2^{|\sigma|}\mu([\sigma] \cap C)$.

A substrategy will have as its inputs a finite partial function $\sigma : \mathbb{N} \rightharpoonup 2$, $n, K, j \in \mathbb{N}$, disjoint finite sets $P, T \subseteq \{0, \ldots, n\}$, and $(t, l) \in \mathbb{N} \times \mathbb{N}$. It will be denoted by $\mathsf{Strat}(\sigma, n, P, j, (T, K), (t, l))$. The interpretation of these parameters is as follows:

- σ determines the clopen set inside which we will work;
- n denotes the index of the next martingale to be considered;
- P denotes the set of martingales for which earlier strategies have determined they are partial, and possibly also n, if this substrategy will work with the assumption (CI:↑);
- if $n \in P$ then j indicates the smallest number of computation steps this strategy believes need to be taken before it believes divergence;
- if $n \in T$ then j indicates after how many computation steps f_n is done betting on σ;
- T is the n-type this strategy will use;
- K is an upper bound for the exceptions to the type T (i.e., above K there are no bits with type a proper superset of T); and
- (t, l) is a pair that is sufficiently out there to compute $\widetilde{\text{Av}}_{T\setminus\{n\}}(\sigma)$.

We will assume that the total versions $\tilde{f}_m$, $m \in T \setminus \{n\}$, have been scaled so that $\widetilde{\mathrm{Av}}_{(T\setminus\{n\})}(\sigma) < 2$ (i.e., instead of working with $\tilde{f}_m$ we work with $\lambda_m \tilde{f}_m$ with suitably chosen coefficients λ_m).

The substrategy will also have assigned to it an initial capital I. It will find an extension τ of σ such that $I \cdot \mu([\sigma])/\mu([\tau]) > 2^n$, and then start all consistent strategies $\mathsf{Strat}(\tau, n+1, P', j', (T', K'), (t', l'))$ with initial capitals such that the total used initial capital stays strictly below 1. A consistent strategy here means that $P \subseteq P' \subseteq \{0, \dots, n+1\}$, $T \subseteq T' \subseteq \{0, \dots, n+1\}$, $P' \cap T' = \emptyset$, both j', K' greater than the current stage of the computation, and (t', l') as determined for the computation of $\mathrm{Av}_T(\tau)$.

Now we describe the action of $\mathsf{Strat}(\sigma, n, P, j, (T, K), (t, l))$ towards finding $\tau \succ \sigma$. This is done in cases:

(1) $n \in P$: find τ' as in (CI:↑) believing that any computation that does not converge in j steps does in fact not converge.

(2) $n \in T$: In this step (and the next) only compute everything for s steps, where s is the stage of the construction.

   Since, by assumption, $\widetilde{\mathrm{Av}}_{(T\setminus\{n\})}(\sigma) < 2$, we can find a coefficient $c_{\sigma,\dots,(t,l)}$ such that if we use $c_{\sigma,\dots,(t,l)} \cdot \tilde{f}_n$ instead of $\tilde{f}_n$ we have that $\widetilde{\mathrm{Av}}_T(\sigma) < 2$, and in fact using j, we can find this coefficient effectively. To simplify notation we write $\tilde{f}_n$ for $c_{\sigma,\dots,(t,l)}\tilde{f}_n$, i.e., in our notation we ignore the coefficient.

   Find a $k > \mathrm{dom}(\sigma)$ such that k is of type T. In searching for this, you find (t', l') that allows you to do the computation to check that k is of this type T (that is, l' is greater than all bets made, and t' is how many computation steps it took for all martingales $\tilde{f}_m$, $m \in T$, to bet on bit k. Then with the input to this strategy, we believe $t'' = \max(t, t', j)$ is sufficiently large to compute both $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 0)\})$ and $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 1)\})$, and from t'' we can compute an appropriate l'' — i.e., (t'', l'') is sufficiently out there to compute both these Av_T.

   Then set τ' to be $\sigma \cup \{(k, i)\}$ for whichever $i \in \{0, 1\}$ gives the least value for $\mathrm{Av}_T(\sigma \cup \{(k, i)\})$.

(3) $n \notin T \cup P$: Find a $k > \mathrm{dom}(\sigma)$ such that k is of type T. In searching for this, you find (t', l') that allows you to do the computation to check that k is of this type T. Then with the input to this strategy, we believe $t'' = \max(t, t')$ and $l'' = \max(l, l')$ is sufficiently out there to compute both $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 0)\})$ and $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 1)\})$.

Now, set τ' to be $\sigma \cup \{(k,i)\}$ for whichever $i \in \{0,1\}$ gives the least value for $\mathrm{Av}_T(\sigma \cup \{(k,i)\})$.

Finally, in all cases, extend τ' to a long enough τ by iterating the construction as in step (3).

3.5. **Verification.**

Recursively (but not computably) determine the sequence

$$\langle (\sigma_i, P_i, j_i, (T_i, K_i), (t_i, l_i)) \mid i \in \mathbb{N} \rangle$$

of correct parameters, that is, where all the assumptions as indicated in the previous section are in fact correct. Then it is clear from the construction that the martingale constructed in the previous section succeeds on all reals in $S = \bigcap_{i \in \mathbb{N}} [\sigma_i]$.

It now remains to show that in S there is an injective random real. We see by induction that for all i we have that $\widetilde{\mathrm{Av}}_{T_i}(\sigma_i) < 2$, and that in fact $\widetilde{\mathrm{Av}}_{\{0,\dots,n\}}(\sigma_i) = \widetilde{\mathrm{Av}}_{T_i}(\sigma_i)$.

Suppose that no real in S is injective random. Then $\mathcal{O} := \bigcup_{i \in \mathbb{N}} \{ [\sigma] \mid d_i(\sigma) > \frac{2}{c_{\sigma_i,\dots,(t_i,l_i)}} \}$ is an open cover of S: Let $A \in S$. Then there is an injective martingale (h_i, q_i) that wins on A. This means in particular that there is an n such that $d_i^n(A) > \frac{2}{c_{\sigma_i,\dots,(t_i,l_i)}} + 2$. This in turn means $d_i^k(A) > \frac{2}{c_{\sigma_i,\dots,(t_i,l_i)}}$ for all $k \geq n$, since d_i was assumed to be slow-but-sure. By now choosing m large enough and setting $\sigma = A \restriction m$, we get $d_i(\sigma) > \frac{2}{c_{\sigma_i,\dots,(t_i,l_i)}}$.

Since S is compact we can find a finite subcover $[\nu_1], \dots, [\nu_n]$ of $\mathcal{O}$. Let b be such that for all $i \leq n$, there is a $j < b$ such that $\tilde{d}_j(\nu_i) > 4$. By replacing some ν_i by $\nu_{i_0}, \dots, \nu_{i_k}$ such that $[\nu_i] = \cup_{j \leq k} [\nu_{i_j}]$ we can assume all ν_i have the same length and that this length is $|\sigma_h|$ for some $h > b$. Also, by the last observation of the previous paragraph, we have that for each a there is a $j < b$ such that $d_j(\nu_a) > \frac{2}{c_{\sigma_j,\dots,(t_j,l_j)}}$. This means that either $\tilde{d}_j(\nu_a) > 2$ or $\widetilde{\mathrm{Av}}_{0,\dots,j-1}(\nu_a) > 2$. Now we have reached the desired contradiction: On the one hand we know $\mathrm{Av}_{\{0,\dots,b\}}(\sigma_h) \leq \mathrm{Av}_{\{0,\dots,h\}}(\sigma_h) < 2$; on the other we have found an antichain covering $[\sigma_h]$ where on each element in the antichain for some $j \leq b$ either some $\tilde{d}_j$ is greater than 2 or $\widetilde{\mathrm{Av}}_{\{0,\dots,j-1\}}$ is greater than 2. This implies that the average $\widetilde{\mathrm{Av}}_{\{0,\dots,b\}}(\sigma_h) > 2$.

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