ISOMORPHISM TYPES OF MAXIMAL COFINITARY GROUPS

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Abstract. A cofinitary group is a subgroup of Sym($\mathbb{N}$) where all nonidentity elements have finitely many fixed points. A maximal cofinitary group is a cofinitary group, maximal with respect to inclusion. We show that a maximal cofinitary group cannot have infinitely many orbits. We also show, using Martin's Axiom, that no further restrictions on the number of orbits can be obtained. We show that Martin's Axiom implies there exist locally finite maximal cofinitary groups. Finally we show that there exists a uniformly computable sequence of permutations generating a cofinitary group whose isomorphism type is not computable.

§1. Introduction. In this paper we study (maximal) cofinitary groups. These are subgroups of the full symmetric group on $\mathbb{N}$, and therefore they have a natural action on $\mathbb{N}$. The structure of (maximal) cofinitary groups has received a lot of attention (see e.g. Adeleke [A], Truss [T1], Brendle, Spinas, and Zhang [BSZ], Hrušák, Steprāns, and Zhang [HSZ], etc.). For a general survey of cofinitary groups see Cameron [C]. Koppelberg [K] has some constructions of cofinitary groups as well. The results we prove here were inspired by two results (Theorem 3 and Theorem 4 below) on maximal cofinitary groups. First we give the definitions.

Definitions 1. (i). Sym(A) is the group of bijections $A \to A$, with the group operation given by composition. We write Id for the identity in this group.

(ii). $f \in \mathrm{Sym}(A)$ is *cofinitary* iff either it has only finitely many fixed points, or it is the identity.

(iii). $G \leq \mathrm{Sym}(A)$ is *cofinitary* iff all $g \in G$ are cofinitary.

(iv). $G \leq \mathrm{Sym}(A)$ is a *maximal cofinitary group* iff G is a cofinitary group and there does not exist a cofinitary group in which G is properly contained.

(v). $G \leq \mathrm{Sym}(A)$ acts on A by the action $(f, a) \mapsto f(a)$.

The usual setting for these definitions is where $A = \mathbb{N}$, and unless we make it clear in the context we will always assume that to be the case.

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Some of the interest in cofinitary groups derives from the fact that they are groups in which all members are eventually different. $f, g \in {}^{\mathbb{N}}\mathbb{N}$ are *eventually different* (almost disjoint) iff

$$\exists k \in \mathbb{N}\ \forall l \geq k\ f(l) \neq g(l).$$

To see that a cofinitary group is an eventually different family, let f, g be members of the group, then fixed points n of $g^{-1}f$ correspond to numbers n such that $f(n) = g(n)$. That maximal cofinitary groups exist follows from Zorn's Lemma: if $\langle G_\alpha \mid \alpha < \beta \rangle$ is an $\subseteq$-increasing chain of cofinitary groups, then $\bigcup_{\alpha<\beta} G_\alpha$ is a cofinitary group that is a $\subseteq$-upper bound for the chain (being cofinitary is a local property).

Often objects that use the axiom of choice in their existence proof do not have nice descriptions from a descriptive set theory point of view. As a relevant example, Mathias [M1] has shown that there is no analytic maximal almost disjoint family (a maximal almost disjoint family (mad family) is an infinite $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ such that for all $X, Y \in \mathcal{A}$ the intersection $X \cap Y$ is finite, and for all infinite $X \subseteq \mathbb{N}$ there is a $Y \in \mathcal{A}$ such that $X \cap Y$ is infinite). Miller [M2] showed an upper bound on the least possible complexity of maximal almost disjoint families; he showed that under the axiom of constructibility there exists a coanalytic maximal almost disjoint family.

Since the definitions of mad families and maximal cofinitary groups are so closely related, this leads to the question

Question 2. What are the possible complexities of maximal cofinitary groups?

Using Miller's method Gao and Zhang proved the following theorem.

Theorem 3 (Gao and Zhang [GZ]). *Under the axiom of constructibility there exists a maximal cofinitary group with a coanalytic generating set.*

We showed that there was enough flexibility in Miller's method to improve this result to the following theorem.

Theorem 4 (Kastermans [K2]). *Under the axiom of constructibility there exists a coanalytic maximal cofinitary group.*

A reasonable question can be how much of an improvement this really is. First note that if the generating set is coanalytic (Π^1_1) then the group itself is at most Σ^1_2. Also Blass (see [GZ]) has observed that if a maximal cofinitary group G is generated by a Σ^1_m set of permutations, then G is Δ^1_m.

Motivated by the two theorems above Anatoly Vershik suggested a generalization of the question of how these two theorems are related. That is he asked how in general the complexity of a cofinitary group relates to the complexity of generating sets for it. Concretely he asked the following question.

Question 5. Does there exist a uniformly computable sequence of cofinitary permutations such that the group they generate does not have a computable isomorphism type (there is no computable group isomorphic to this group)?

Also Question 2 has now been reduced to the following

Question 6. Does there exist a maximal cofinitary group that is Borel as a subset of ${}^{\mathbb{N}}\mathbb{N}$?

A relevant observation about the proofs of the results above, and many earlier constructions of maximal cofinitary groups, is that the constructions always give rise to free groups (this method of construction using so called good extensions can be seen for instance in Zhang [Z1] and Zhang [Z2]). The constructions are performed by recursively adding new generators, and these generators are free over the group that is already constructed.

Let $G \leq \mathrm{Sym}(\mathbb{N})$, then we say $g \in \mathrm{Sym}(\mathbb{N}) \setminus G$ is *free over* G iff $\langle G, g\rangle$ is isomorphic to $G * F(x)$ by the map $f : G * F(x) \to \langle G, g\rangle$ that is the identity on G and maps x to g (here $F(x)$ is the free group with generator x, and $G * H$ denotes the free product of the groups G and H).

It seems to us easily imaginable that the least complexity of maximal cofinitary groups and the least complexity of freely generated maximal cofinitary groups are not equal. This leads to the following question

Question 7. What are the possible isomorphism types of maximal cofinitary groups?

Here we show (see Section 5) the following

Theorem 8. *Martin's axiom implies that there exists a locally finite maximal cofinitary group.*

Related to the isomorphism type of a group is how it can act, or what its orbit structure is. Here an orbit $O \subseteq \mathbb{N}$ is a set on which the action is transitive and which is closed under the action. This is also related to Question 6. We show the following two theorems (see Sections 3 and 4, from [K1])

Theorem 9. *A cofinitary group with infinitely many orbits is not maximal.*

Theorem 10. *Martin's axiom implies that for any $n, m \in \mathbb{N}$ with $n \neq 0$ there exists a maximal cofinitary group with n infinite orbits and m finite orbits.*

These determine the possible sizes and numbers of orbits of the action on the natural numbers. The following question is still open, where the diagonal action is defined by $(g, (n_0, \dots, n_{k-1})) \mapsto (g(n_0), \dots, g(n_{k-1}))$.

Question 11. Does there exist a maximal cofinitary group that has infinitely many orbits on $\mathbb{N}^k$ $(k > 1)$ under the diagonal action?

If the answer to this question is no, than we get information on Question 6. It is still open if maximal cofinitary groups can be closed. A closed group with only finitely many orbits in $\mathbb{N}^k$ for any $k \in \mathbb{N}$ is called oligomorphic. Also any closed subgroup of $\mathrm{Sym}(\mathbb{N})$ is the isomorphism group of a countable structure. If the closed group is oligomorphic, then this countable structure is $\aleph_0$-categorical. Since much is known about $\aleph_0$-categorical structures, this might help in deciding if closed maximal cofinitary groups exist.

We then return to Question 5 to show (see Section 6) how changing the method of construction for maximal cofinitary groups also allows us to answer it positively. Part of this change in the method was introduced in Kastermans [K1] to show that from the continuum hypothesis one can construct a maximal cofinitary group into which every countable group embeds. In our current context some of the proof simplifies because we are not extending a given countable group but constructing the whole group at once.

A construction of a cofinitary group into which every countable group embeds was already performed by Truss [T1] and Adeleke [A] and further analysed in Truss [T2]. Those analyses did not include the notions of good extensions in the context of a group that is not free.

After being told about Theorem 9 Blass observed the following first restriction (other than size restrictions) on the isomorphism type of maximal cofinitary groups.

Theorem 12 (Blass). *No Abelian group has an action that is maximal cofinitary.*

Proof. Let G be an Abelian group. Since no countable group has a maximal cofinitary action we can find $\langle e_i \mid i < \kappa \rangle$, $\kappa \geq \aleph_1$ and $e_i \neq e_j$ for $i \neq j$, generating G.

If the action of G is maximal cofinitary it will have an infinite orbit $A \subseteq \mathbb{N}$. There is some $\beta < \kappa$ such that A is already an orbit of the group generated by $\{e_i \mid i < \beta\}$. Pick $a \in A$. Any e_j, $j \geq \beta$, is completely determined by where it maps a. Since there are only countably many choices for where to map a there are distinct $j, k \geq \beta$ such that $e_j \restriction A = e_k \restriction A$. This contradicts that the action is cofinitary. $\dashv$

§2. Basics, notation, and conventions. In this section we establish some notation and conventions. We also make some very basic observations. Note again that we will assume throughout that $A = \mathbb{N}$ unless specifically mentioned otherwise.

First we identify function $f : A \to B$ with its graph, which means in particular that $(a, b) \in f$ is equivalent to $f(a) = b$.

For X any set we write $F(X)$ for the free group with generators X, and if $X = \{x\}$ we write $F(x)$ for that group.

If G and H are groups, we write $G * H$ for their free product.

$[A]^\omega$ denotes the collection of infinite subsets of A.

If G is a group and X a set, we write $W_{G,X}$ for $G*F(X)$, and if $X = \{x\}$ we write W_G for $W_{G,\{x\}}$. We identify $W_{G,X}$ with the set of reduced words; for instance each $w(x) \in W_G$ is of the form $g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_l} g_{l+1}$ where $g_i \in G$ with $g_i \neq \mathrm{Id}$ for $1 \leq i \leq l$, and $k_i \in \mathbb{Z} \setminus \{0\}$ for all i such that $0 \leq i \leq l$.

If $G \leq \mathrm{Sym}(A)$, $w \in W_{G,X}$, and $\vec{g} = \langle g_x \mid x \in X \rangle \subseteq \mathrm{Sym}(A)$, then $w(\vec{g})$ is the image of w under the map $f : W_G \to \langle G, \vec{g} \rangle$ defined to be the identity of G and mapping x to g_x for all $x \in X$. Note that for $w \in W_G$, $w(g) = g_0 g^{k_0} g_1 g^{k_1} \cdots g^{k_l} g_{l+1}$ if $w(x) = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_l} g_{l+1}$ and that similar statements are true for all $W_{G,X}$.

For $w(x) = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_l} g_{l+1} \in W_G$ we define the *length* $\mathrm{lh}(w)$ of w to be $l+1+\sum_{i \leq l} k_i$. For $i \leq \mathrm{lh}(w)$ we define $\mathrm{oc}(w, i)$ to be the i-th symbol in w counted from the right; e.g. if $w(x) = g_0 x^{-2} g_1$, then $\mathrm{oc}(w, 0) = g_1$, $\mathrm{oc}(w, 1) = x^{-1}$, $\mathrm{oc}(w, 2) = x^{-1}$, and $\mathrm{oc}(w, 3) = g_0$.

We write $f : A \rightharpoonup B$ for a partial map f from A to B.

If $\vec{f} = \langle f_x : A \rightharpoonup A \mid x \in X \rangle$, $w \in W_{G,X}$, and $n \in A$ we define the *evaluation path* of n in $w(\vec{f})$ to be $\langle z_i \mid i \leq j \rangle$ with $z_0 = n$,

$$z_{i+1} = \begin{cases} f_x(z_i) & \text{if } \mathrm{oc}(w, i) = x, \\ f_x^{-1}(z_i) & \text{if } \mathrm{oc}(w, i) = x^{-1}, \\ \mathrm{oc}(w, i)(z_i) & \text{otherwise } (\mathrm{oc}(w, i) \in G). \end{cases}$$

and either $z_j \notin \mathrm{dom}(f_x)$ (if $\mathrm{oc}(w, i) = x$), $z_j \notin \mathrm{ran}(f_x)$ (if $\mathrm{oc}(w, i) = x^{-1}$), or $j = \mathrm{lh}(w)$. We say the evaluation path is *total* in the second case (i.e. $j = \mathrm{lh}(w)$), otherwise we call it *partial.*

Now for $\vec{f}$ and $w \in W_{G,X}$, we define $w(\vec{f})$ to be $\{(n, k) \in A \times A \mid$ there is an evaluation path $\vec{z}$ for $\vec{f}$ in w of length $\mathrm{lh}(w)$ such that $z_0 = n$ and $z_{\mathrm{lh}(w)} = k\}$. Note that this corresponds to the definition of $w(\vec{g})$ above in case $\vec{g}$ is a sequence of permutations.

§3. No maximal cofinitary group has infinitely many orbits. Here we prove the following theorem.

Theorem 13. *A cofinitary group with infinitely many orbits is not maximal.*

Suppose G is a cofinitary group with infinitely many orbits. Fix an enumeration without repetitions $\langle O_i \mid i \in \mathbb{N}\rangle$ of all orbits of G. From these data we define a function h such that $h \notin G$ and $\langle G, h\rangle$ is cofinitary, showing G is not maximal. We will first define h, show some of its properties and finally show how these properties can be used to show that h is as required.

We define $h : \mathbb{N} \to \mathbb{N}$ by a sequence of finite approximations h_s, $s \in \mathbb{N}$. Set $h_0 := \emptyset$, and suppose h_s has been defined. Let $n := \min\big((\mathbb{N} \setminus \mathrm{dom}(h_s)) \cup (\mathbb{N} \setminus \mathrm{ran}(h_s))\big)$ and $m := \min O_j$, where j is the least number such that $O_j \cap \big(\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s)\big) = \emptyset$. Then set $h_{s+1} := h_s \cup \{(n, m)\}$ if $n \notin \mathrm{dom}(h_s)$ and $h_{s+1} := h_s \cup \{(m, n)\}$ otherwise.

Clearly $h \in \mathrm{Sym}(\mathbb{N}) \setminus G$; we only need to verify that for all $w(x) \in W_G$ the function $w(h)$ has finitely many fixed points, or is the identity. It will in fact be the case that all $w(h)$ (except the identity word) have only finitely many fixed points; showing this will take some work.

First note that for all O_i and O_j there is at most one pair $(a, b) \in h$ such that $a \in O_i$ and $b \in O_j$. But in fact *much* more is true. This much more is described by the following definition, which also describes the picture from which this proof developed.

Definition 14. The *G-orbits tree of h* has vertex set $\{O_j \mid j \in \mathbb{N}\}$. It has an edge between O_j and O_i if there is an $n \in O_j$ such that $h(n) \in O_i$.

We need to see that this defines a tree. Suppose not, then there is a cycle $O_{n_0}, O_{n_1}, \ldots, O_{n_l} = O_{n_0}$ and for all $0 \le i < l$ vertex O_{n_i} is connected to vertex $O_{n_{i+1}}$. This means that for every $0 \le i < l$ there is a pair $(a, b) \in h$ such that $a \in O_{n_i}$ and $b \in O_{n_{i+1}}$ or $a \in O_{n_{i+1}}$ and $b \in O_{n_i}$. By the observation above these pairs are unique. Let $s \in \mathbb{N}$ be the least s such that all pairs $\langle a, b\rangle$ used in this cycle are in h_{s+1}. Since s is least with this property the unique pair $\langle a, b\rangle \in h_{s+1} \setminus h_s$ is used in the cycle. Then $\langle a, b\rangle$ connects some O_{n_j} with one of its neighbors, $O_{n_{j-1}}$ or $O_{n_{j+1}}$. But each of these is already connected to its other neighbor, so a lies in some O_k and b lies in some O_l such that both $O_k \cap (\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s)) \neq \emptyset$ and $O_l \cap (\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s)) \neq \emptyset$. This however means that the pair (a, b) does not satisfy the defining criterion for inclusion in h_{s+1}; so we have the contradiction we were looking for.

The next definition gives us a way to talk about the process of evaluating a word $w(h)$ on a number n. The orbit path defined here can be looked at as a walk on the vertices of the G-orbit tree of h.

Definition 15. For $m \in \mathbb{N}$, $w(x) = g_0 x^{k_0} g_1 \cdots x^{k_{l-1}} g_l \in W_G$ and $h \in \mathrm{Sym}(\mathbb{N})$ we define the *orbit path* of n in $w(h)$ to be the sequence of orbits the evaluation passes through — that is $\vec{l} = \langle l_i \mid 0 \leq i \leq \mathrm{lh}(w)\rangle$ where $l_i = j$ iff $z_i \in O_j$ with $\vec{z}$ the evaluation path for w on n.

One of the essential features of the function h we have defined is that for any $n \in \mathbb{N}$ and $w(x) \in W_G$, the evaluation path for n and the orbit path of n determine each other. This equivalence will be useful as in a word with infinitely many fixed points the action on the orbits allows us to conclude that one of the g_i in $w(x)$ has infinitely many fixed points which will be the conclusion, at that time desired.

We are now ready to finish the proof of Theorem 13. Suppose, towards a contradiction, that there is a $w \in W_G$ such that $w(h)$ has infinitely many fixed points. We will show that each fixed point of $w(h)$ gives rise to a fixed point in some g_i appearing in w.

Let n be one of the fixed points of $w(h)$ and $\vec{l}$ its orbit path. Let $l_i \in \vec{l}$ be such that O_{l_i} is the first vertex realizing the maximal distance from O_{l_0} in the G-orbit tree of h. The orbit $O_{l_{i-1}}$ preceding O_{l_i} is closer to O_{l_0}, so $w_{(i)} = x$ or x^{-1} (application of any member of G will not change the orbit we are in) and $(w \restriction i)(n) \in O_{l_{i-1}}$ and $(w \restriction i+1)(n) \in O_{l_i}$ with $((w \restriction i)(n), (w \restriction i+1)(n)) \in h$ or $((w \restriction i+1)(n), (w \restriction i)(n)) \in h$.

Assume the former; the other case is analogous.

Since the G-orbit tree of h is a tree, $O_{l_{i-1}}$ and O_{l_i} are connected by an edge and $O_{l_{i-1}}$ is strictly closer to O_{l_0} than O_{l_i}, all the other neighbors of O_{l_i} are strictly closer to O_{l_0} than O_{l_i}. This means that the first vertex after O_{l_i} different from O_{l_i} has to be equal to $O_{l_{i-1}}$. But as $((w \restriction i)(n), (w \restriction i+1)(n))$ is the only pair in h allowing direct passage between $O_{l_{i-1}}$ and O_{l_i} this means we have to apply h^{-1} with input $(w \restriction i+1)(n)$ to get back to $O_{l_{i-1}}$.

We have the following situation in the orbit path of n:

$$\begin{aligned}(w \restriction i)(n) \in O_{l_{i-1}} &\xrightarrow{h} (w \restriction i+1)(n) \in O_{l_i} \to \cdots \\ &\to (w \restriction i+i)(n) \in O_{l_i} \xrightarrow{h^{-1}} (w \restriction i)(n) \in O_{l_{i-1}}\end{aligned}$$

Now, in between arriving at O_{l_i} and leaving O_{l_i} we obviously stay in the same orbit. This means that between arriving at O_{l_i} and leaving we can only apply members of G. By the shape of w we apply exactly one member g_i of G. And by the work above this member has to fix $(w \restriction i+1)(n)$.

We now know that every fixed point of $w(h)$ gives rise to a fixed point in some g_j appearing in w. There is therefore a j such that infinitely many fixed points of $w(h)$ give rise to a fixed point in that g_j. No two such fixed points of $w(h)$ can be associated to the same fixed point of g_j as for different points the jth members of their respective evaluation paths are never equal. (Note that here we strictly speaking need to consider the pair (g_j, j) consisting of the group element together with an indication of where it occurs in the word. It is possible that one group element occurs more than once.) This shows that this g_j appearing in w has infinitely many fixed points, contradicting that it is a member of the cofinitary group G.

§4. A maximal cofinitary group with finitely many infinite orbits. In this section we prove the following theorem.

THEOREM 16. *Martin's axiom implies that for every* $n \in \mathbb{N} \setminus \{0\}$ *and* $m \in \mathbb{N}$ *there exists a maximal cofinitary group with exactly* n *infinite orbits and exactly* m *finite orbits.*

We use some machinery from Gao and Zhang [GZ].

DEFINITIONS 17. (i). Let $p, q : A \rightharpoonup A$ be finite partial injective functions, and $w \in W_G$. Then q is a *good extension* of p with respect to w iff $p \subseteq q$ and for every $n \in \mathbb{N}$ such that $w(q)(n) = n$ there exist $l \in \mathbb{N}$, and $u, z \in W_G$ such that
- $w = u^{-1}zu$ without cancellation,
- $z(p)(l) = l$, and
- $u(q)(n) = l$.

Note that if $w(p)(n) = n$ we can choose $z = w$ and $u = \mathrm{Id}$.

With these definitions the following lemmas can be proved as was done in Gao and Zhang [GZ], or can be derived from the proofs there using a bijection $\mathbb{N} \to A$.

LEMMA 18. *Let* $G \leq \mathrm{Sym}(A)$ *for some* $A \in [\mathbb{N}]^\omega$, $p : A \rightharpoonup A$ *finite and injective, and* $w \in W_G$. *Then*
- *(Domain Extension Lemma) For each* $n \in A \setminus \mathrm{dom}(p)$, *for all but finitely many* $k \in A$, *the extension* $p \cup \{(n, k)\}$ *is a good extension of* p *with respect to* w.
- *(Range Extension Lemma) For each* $k \in A \setminus \mathrm{ran}(p)$, *for all but finitely many* $n \in A$, *the extension* $p \cup \{(n, k)\}$ *is a good extension of* p *with respect to* w.

DEFINITION 19. For $A \in [\mathbb{N}]^\omega$ and $G \leq \mathrm{Sym}(A)$ a partial function $f : A \rightharpoonup A$ is *hitable* with respect to G if
- for all $g \in G$ $f \setminus g$ is infinite, and
- for all $w \in W_G$ either $w(f) \cong \mathrm{Id}$ ($w(f)$ is the identity where defined), or $w(f)$ has only finitely many fixed points.

Note that for f total and $G \leq \mathrm{Sym}(A)$ a cofinitary group, f being hitable w.r.t. G means that $f \in \mathrm{Sym}(A) \setminus G$ and $\langle G, f \rangle$ is cofinitary. Also note that if f is hitable with respect to G and $f' \subseteq f$ is infinite, then f' is hitable with respect to G as well (to see the first clause, observe that with the second clause it follows that $f \cap g$ is finite). Lastly note that if an infinite f is not hitable with respect to G then there exists a $g \in G$ such that $f \cap g$ is infinite; this means that either $f \subseteq g$ or any extension of f to a total function $\bar{f}$ has that $g^{-1}\bar{f}$ has infinitely many fixed points but is not the identity.

LEMMA 20 (Hitting f Lemma). *Let $A \in [\mathbb{N}]^\omega$, $G \leq \mathrm{Sym}(A)$ a cofinitary group, $p : A \rightharpoonup A$ finite injective, and $w \in W_G$. If $f : A \rightharpoonup A$ is hitable with respect to G, then there exists $n \in \mathrm{dom}(f)$ such that $p \cup \{(n, f(n))\}$ is a good extension of p with respect to w.*

The proof of [KZ, Lemma 15] works to show this.

Now we give three applications of Martin's axiom that will be used repeatedly in the construction of the maximal cofinitary group. First we define the poset we use.

DEFINITION 21. Let $A \in [\mathbb{N}]^\omega$ and $G \leq \mathrm{Sym}(A)$. Define $\mathbb{P}_G$ as follows:

- $\mathbb{P}_G = \{\langle p, W\rangle \mid p : A \rightharpoonup A$ finite and injective, and $W \subseteq W_G$ finite$\}$.
- $\langle q, W_q\rangle \leq_{\mathbb{P}_G} \langle p, W_p\rangle$ iff $p \subseteq q$, $W_p \subseteq W_q$, and q is a good extension of p with respect to all words in W_p.

$\mathbb{P}_G$ fulfils the countable chain condition, as any two elements with identical first coordinate are compatible.

The Domain Extension Lemma shows that $D_n = \{\langle p, W\rangle \in \mathbb{P}_G \mid n \in \mathrm{dom}(p)\}$ is dense for all $n \in A$, and the Range Extension Lemma shows that $R_n = \{\langle p, W\rangle \in \mathbb{P}_G \mid n \in \mathrm{dom}(p)\}$ is dense for all $n \in A$. The set $W_w = \{\langle p, W\rangle \in \mathbb{P}_G \mid w \in W\}$ is also dense for all $w \in W_G$.

Construction 1: construction of a new element. Let $T \in [\mathbb{N}]^\omega$ and $G \leq \mathrm{Sym}(T)$ be a cofinitary group such that $|G| < \mathfrak{c}$. Then there exists $g \in \mathrm{Sym}(T)$ such that $\langle G, g\rangle$ is cofinitary, and $\langle G, g\rangle \cong G * \langle g\rangle \cong G * F(x)$.

Let $\mathcal{G} \subseteq \mathbb{P}_G$ be a filter such that $\mathcal{G} \cap D \neq \emptyset$ for all dense sets D in $\{D_n \mid n \in T\} \cup \{R_n \mid n \in T\} \cup \{W_w \mid w \in W_G\}$ (and possibly others). That such a $\mathcal{G}$ exists follows from Martin's axiom. Then $g = \bigcup_{(p,W) \in \mathcal{G}} p$ is as required: since $\mathcal{G}$ is a filter, all finite injective partial functions in it are compatible, since $\mathcal{G} \cap D_n \neq \emptyset$ for all $n \in T$ the domain is all of T and since $\mathcal{G} \cap R_n \neq \emptyset$ for all $n \in T$ the range is all of T. To see that $\langle G, g\rangle$ is cofinitary and isomorphic to $G * F(x)$ we need to see that for all $w \in W_G$ the permutation $w(g) : T \to T$ has only finitely many fixed points. Let V be the finite set of all subwords (not necessarily proper) of w. Since W_v is dense for all $v \in V$ there exists $\langle p, W\rangle \in \mathcal{G}$ such that $V \subseteq W$. Let z be the shortest subword of w conjugate to w, then $w(g)$ has the same number of fixed points as $z(p)$ since in the definition of good extension with respect to w or a conjugate subword we can always choose the conjugate subword to be z, and then any fixed point in $w(g)$ has to come from a fixed point of $z(p)$.

Construction 2: construction of a new element with respect to to an infinite set. Let $T \in [\mathbb{N}]^\omega$, $S \in [T]^\omega$, and $G \leq \mathrm{Sym}(T)$ be a cofinitary group such that $|G| < \mathfrak{c}$. Then there exists a $g \in \mathrm{Sym}(T)$ such that $\langle G, g\rangle$ is cofinitary, $\langle G, g\rangle \cong G * \langle g\rangle \cong G * \mathbb{Z}$, and $g \cap (S \times S)$ is infinite.

For this note that $S_n = \{\langle p, W\rangle \in \mathbb{P}_G \mid \exists k \geq n\; k \in S \cap \mathrm{dom}(p) \wedge p(k) \in S\}$ is dense for each $n \in T$. This follows since if $\langle p, W\rangle \in \mathbb{P}_G$, then there is a $k \in S \setminus \mathrm{dom}(p)$ with $k \geq n$.. By the Domain Extension Lemma for all but finitely many m the extension $p \cup \{(k, m)\}$ is a good extension of p with respect to all words in W, therefore there exists such an m that is a member of S.

Adding these countably many sets to the dense sets in Construction 1, and taking a filter $\mathcal{G}$ intersect all these as well gives us the required element.

Construction 3: construction of a new element with respect to a hitable function. Let $T \in [\mathbb{N}]^\omega$, $G \leq \mathrm{Sym}(T)$ cofinitary such that $|G| < \mathfrak{c}$, and $f : T \rightharpoonup T$ hitable with respect to G. Then there exists $g \in \mathrm{Sym}(T)$ such that $\langle G, g\rangle$ is cofinitary, $\langle G, g\rangle \cong G * \langle g\rangle \cong G * F(x)$, and $g \cap f$ is infinite.

For this note that for each $n \in T$, the set $F_n = \{\langle p, W\rangle \in \mathbb{P}_G \mid (\exists k \geq n)\; k \in \mathrm{dom}(p) \;\wedge\; f(k) = p(k)\}$ is dense by the Hitting f Lemma. Adding these countably many sets to the dense sets in Construction 1, and taking a filter $\mathcal{G}$ which intersects all these gives us the required element.

Construction of the group. Now let $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{N}$ be given. We need to construct a maximal cofinitary group with n infinite orbits and m finite orbits. Choose a partition of $\mathbb{N}$ into $\bigcup_{i<n} O_i \cup \bigcup_{i<m} O'_i$ with $O_i \subseteq \mathbb{N}$ all infinite, and $O'_i \subseteq \mathbb{N}$ all finite.

We are going to construct sequences of generators $\vec{g}_i = \langle g_{i,\alpha} \in \mathrm{Sym}(O_i) \mid \alpha < \mathfrak{c}\rangle$, and $\vec{g}'_i = \langle g'_{i,\alpha} \in \mathrm{Sym}(O'_i) \mid \alpha < \mathfrak{c}\rangle$ such that $\langle \vec{g}_i\rangle$ is transitive on O_i and is freely generated by $\vec{g}_i$, and $\langle \vec{g}'_i\rangle$ is transitive on O'_i.

If we have all these sequences constructed up to length α we define $G_{i,\alpha}$ to be the group generated by $\vec{g}_i$, $G'_{i,\alpha}$ the group generated by $\vec{g}'_i$, and the group G_α to be generated by g_β $(\beta < \alpha)$ where g_β is defined as follows

$$g_\beta(x) = \begin{cases} g_{i,\beta}(x), & \text{if } x \in O_i; \\ g'_{i,\beta}(x), & \text{if } x \in O'_i. \end{cases}$$

We will construct the sequences so that $G_\mathfrak{c}$ is a maximal cofinitary group.

For each O'_i we choose a permutation $g_{i,0}$ in $\mathrm{Sym}(O'_i)$ such that $\langle g_{i,0}\rangle$ is transitive, and set $g_{i,\alpha} = g_{i,0}$ for all $\alpha < \mathfrak{c}$.

The permutation

$$h(x) = \begin{cases} x+2, & \text{if } x \text{ is even;} \\ 0, & \text{if } x = 1; \\ x-2, & \text{otherwise.} \end{cases}$$

generates a cofinitary transitive group on $\mathbb{N}$. Conjugating by a bijection between O_i and $\mathbb{N}$ we get a permutation in $\mathrm{Sym}(O_i)$ that generates a transitive and cofinitary group; let $g_{i,0}$ be that permutation.

Now enumerate $\mathrm{Sym}(\mathbb{N})$ by $\langle f_\alpha \mid 0 < \alpha < \mathfrak{c}\rangle$. In our recursive construction at step α we will construct $g_{i,\alpha}$ for all $i < n$. If $\langle f_\alpha, G_\alpha\rangle$ is not cofinitary, construct all $g_{i,\alpha}$ using Construction 1.

Otherwise since f_α is a permutation there are $i, j < n$ such that $f_\alpha \cap (O_i \times O_j)$ is infinite. Choose such i and j. If $i = j$, then construct $g_{i,\alpha}$ using Construction 3 ($f_\alpha \cap (O_i \times O_i)$ is hitable with respect to $G_{i,\alpha}$ since $\langle f_\alpha, G_\alpha\rangle$ is cofinitary). For all $l \neq i$ construct $g_{l,\alpha}$ using Construction 1.

If $i \neq j$ first construct $g_{j,\alpha}$ by Construction 2 using the infinite set $\mathrm{ran}(f_\alpha \cap (O_i \times O_j))$. Then look at the function $(f_\alpha \cap (O_i \times O_j))^{-1} g_{j,\alpha} (f_\alpha \cap (O_i \times O_j))$. This is an infinite partial function $O_i \rightharpoonup O_i$. If it is not hitable with respect to $G_{i,\alpha}$ construct $g_{i,\alpha}$ using Construction 1, otherwise construct it using Construction 3. Construct all $g_{l,\alpha}$ for $i \neq l \neq j$ using Construction 1.

This completes the construction. We need to check that the group constructed is as we want it:

All the O_i and O'_i are indeed orbits as all elements map elements in one of these sets to the same set, and the first element of each sequence of generators generates a group that is transitive on the corresponding set.

It is easy to check by induction that each G_α is cofinitary, and therefore the group $G_\mathfrak{c}$ is cofinitary. The only property that remains to be checked is the maximality. So let $f \in \mathrm{Sym}(\mathbb{N})$. Then $f = f_\alpha$ for some α such that $0 < \alpha < \mathfrak{c}$. If $\langle G_\mathfrak{c}, f_\alpha\rangle$ is cofinitary, but $f_\alpha \notin G_\mathfrak{c}$, then the same is true with respect to G_α. Then in step α we either construct a $g_{i,\alpha}$ such that $g_{i,\alpha} \cap f_\alpha$ is infinite, or we construct a $g_{j,\alpha}$ such that $f_\alpha^{-1} g_{j,\alpha} f_\alpha$ has generates an element with infinitely many fixed points when added to $G_{i,\alpha}$, or we construct a $g_{j,\alpha}$ and $g_{i,\alpha}$ such that $g_{i,\alpha} \cap f_\alpha^{-1} g_{j,\alpha} f_\alpha$ is infinite. All three cases imply that either $f_\alpha \in G_{\alpha+1}$ or $\langle f_\alpha, G_{\alpha+1}\rangle$ is not cofinitary, as was to be shown.

§5. A locally finite maximal cofinitary group. In this section we prove the following theorem.

THEOREM 22. *Martin's axiom implies that there exists a locally finite maximal cofinitary group.*

So let G be a locally finite cofinitary group such that $|G| < \mathfrak{c}$, and let $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ be such that $\langle G, f\rangle$ is cofinitary. To prove the theorem it suffices to find $g \in \mathrm{Sym}(\mathbb{N})$ such that $g \cap f$ is infinite, and $\langle G, g\rangle$ is a locally finite cofinitary group.

If H is a group then any action isomorphic to the action of H on itself is called a *regular action*. Any action is regular if it is transitive and no member other than the identity has fixed points. An action is *semiregular* if the stabilizer of any point is trivial. So an action is regular if it is semiregular and transitive.

We first prove the following lemma that provides one of the main ingredients of the construction.

Lemma 23. *Let*

- $H \leq \mathrm{Sym}(\mathbb{N})$ *be finite,*
- $A, B \subseteq \mathbb{N}$ *be finite sets such that H acts regularly on B, H acts on $A \cup B$, and H acts semiregularly on $\mathbb{N} \setminus A \cup B$,*
- $f \in \mathrm{Sym}(A \cup B)$.

Then there exists finite $C \subseteq \mathbb{N} \setminus (A \cup B)$ and $\bar{f} \in \mathrm{Sym}(A \cup B \cup C)$ such that $f \subseteq \bar{f}$, and $\langle H \restriction C, \bar{f} \restriction C\rangle$ acts regularly on C (in particular H acts semiregularly on C).

Proof. The regular action of $\langle H \restriction (A \cup B), f\rangle$ consists of copies of the regular action of $H \restriction (A \cup B)$ (which is the same as the regular action of H). Write $\langle H \restriction (A \cup B), f\rangle = \coprod_{i\in I} H_i$ where $H_i \cong H$ $(i \in I)$ and $I \subseteq \mathbb{N}$ is finite ($\coprod$ denotes disjoint union). Let C be the union of $|I|$ many orbits of H all contained in $\mathbb{N} \setminus (A \cup B)$, and write $C = \coprod_{i\in I} C_i$ where the C_i are H orbits.

Then choose $h_i \in H_i$ $(i \in I)$ and $c_i \in C_i$ $(i \in I)$. Define a bijection $F : \coprod_{i\in I} H_i \to \coprod_{i\in I} C_i$ by mapping $h_i \mapsto c_i$ and extending this using the H action: for each $g \in H_i$ there is an $h \in H$ such that $g = hh_i$. Then $F(g) = hc_i$.

Note that now the action of $\langle H, FfF^{-1}\rangle$ is isomorphic to the regular action of $\langle H \restriction (A \cup B), f\rangle$. So if we set $\bar{f} = f \cup FfF^{-1}$ then $\bar{f}$ and C are as required. $\dashv$

We need the following two lemmas on finite cofinitary groups.

Lemma 24. *If H is a finite cofinitary group, then H acts regularly on all but finitely many of its orbits.*

Proof. Suppose H has infinitely many orbits on which it does not act regularly. Then for each of these orbits there exists an element of H other than the identity that has a fixed point. Since H is finite this means that there exists an element of H with infinitely many fixed points contradicting that it acts cofinitarily. $\dashv$

LEMMA 25. *Let H be a finite cofinitary group, and $f \in \mathrm{Sym}(\mathbb{N}) \setminus H$ such that $\langle H, f\rangle$ is cofinitary. Then for all but finitely many $n \in \mathbb{N}$ the numbers n and $f(n)$ are not in the same H-orbit.*

PROOF. Suppose that for infinitely many n we have that n and $f(n)$ are in the same H-orbit. For each such pair $(n, f(n))$ there exists an element h of H such that $h(n) = f(n)$. Since H is finite there is an element h of H that is used for infinitely many n, but this means $f \cap h$ is infinite. Since $f \notin H$ this is a contradiction with $\langle G, f\rangle$ being cofinitary. $\dashv$

We write $\vec{g} \in G$ iff $\vec{g} = \{g_0, \dots, g_n\}$ and for all $i \leq n$, $g_i \in G$, and $\{\Delta_i^{\vec{g}} \mid i \in \mathbb{N}\}$ for the set of orbits of $\langle \vec{g}\rangle$.

The poset $\mathbb{P}_G$ is defined as follows

- $(s, \vec{g}) \in \mathbb{P}_G$ iff $\vec{g} \in G$ and $s : \mathbb{N} \rightharpoonup \mathbb{N}$ is a finite permutation such that $\mathrm{dom}(s) = \mathrm{ran}(s) = \coprod_{i \in I} \Delta_i^{\vec{g}}$ for $I \subseteq \mathbb{N}$ with $|I| \geq 2$ and $\{\Delta_i^{\vec{g}} \mid i \in I\}$ contains all $\langle \vec{g}\rangle$ orbits on which the $\langle \vec{g}\rangle$ action is not regular, and at least one orbit on which it does act regularly.
- $(s_1, \vec{g}_1) \leq_{\mathbb{P}_G} (s_0, \vec{g}_0)$ iff $s_0 \subseteq s_1$, $\vec{g}_0 \subseteq \vec{g}_1$, and the action of $\langle s_1 \restriction (\mathrm{dom}(s_1) \setminus \mathrm{dom}(s_0)), \vec{g}_0 \restriction (\mathrm{dom}(s_1) \setminus \mathrm{dom}(s_0))\rangle$ consists of copies of the regular action of $\langle s_1, \vec{g}_0 \restriction \mathrm{dom}(s_1)\rangle$.

Define the following subsets of $\mathbb{P}_G$

- $D_n = \{(s, \vec{g}) \in \mathbb{P}_G \mid n \in \mathrm{dom}(s)\}$, for all $n \in \mathbb{N}$,
- $R_n = \{(s, \vec{g}) \in \mathbb{P}_G \mid n \in \mathrm{ran}(s)\}$, for all $n \in \mathbb{N}$,
- $E_g = \{(s, \vec{g}) \in \mathbb{P}_G \mid g \in \vec{g}\}$, for all $g \in G$, and
- $H_{f,n} = \{(s, \vec{g}) \in \mathbb{P}_G \mid \exists k \geq n\ s(k) = f(k)\}$, for all $n \in \mathbb{N}$ and $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f\rangle$ is cofinitary.

We first show that all these sets are dense, and then that this suffices to show the result.

For all $n \in \mathbb{N}$ the set D_n is dense in $\mathbb{P}_G$: Let $(s, \vec{g} \in \mathbb{P}_G$ and suppose $n \notin \mathrm{dom}(s)$ (otherwise $(s, \vec{g}) \in D_n$ and we are done). Write $\mathrm{dom}(s) = \mathrm{ran}(s) = \coprod_{i \in I} \Delta_i^{\vec{g}}$. The regular action of $\langle s, \vec{g} \restriction \mathrm{dom}(s)\rangle$ consists of some finite number k copies of the regular action of $\langle \vec{g} \restriction \mathrm{dom}(s)\rangle$ (which is equal to the regular action of $\langle \vec{g}\rangle$). Write $\langle s, \vec{g} \restriction \mathrm{dom}(s)\rangle = \coprod_{j<k} \langle \vec{g}\rangle$ the decomposition of $\langle s, \vec{g} \restriction \mathrm{dom}(s)\rangle$ into copies of the regular action of $\langle \vec{g}\rangle$.

Choose k elements $i_0, \dots, i_{k-1} \in \mathbb{N} \setminus I$ such that $n \in \Delta_{i_0}^{\vec{g}}$. Let $F : \coprod_{j<k} \langle \vec{g}\rangle \to \bigcup_{l<k} \Delta_{i_l}^{\vec{g}}$ be the bijection respecting the action of $\vec{g}$ (can be obtained as in the lemma above by choosing a point in each orbit and extending according to the action).

Then $\langle FsF^{-1}, \vec{g} \restriction (\bigcup_{l<k} \Delta_{i_l}^{\vec{g}}\rangle$ acting on $\bigcup_{l<k} \Delta_{i_l}^{\vec{g}}$ is a regular action that is an isomorphic copy of the regular action of $\langle s, \vec{g}\rangle$. This shows that $(s \cup FsF^{-1}, \vec{g}) \leq_{\mathbb{P}_G} (s, \vec{g})$. Since $n \in \mathrm{dom}(s \cup FsF^{-1})$ this shows D_n is dense.

For all $n \in \mathbb{N}$ the set R_n is dense in $\mathbb{P}_G$: the proof is similar to that for D_n.

For all $g \in G$ the set E_g is dense in $\mathbb{P}_G$: Let $(s, \vec{g}) \in \mathbb{P}_G$. Write $\mathrm{dom}(s) = \coprod_{i \in I} \Delta_i^{\vec{g}}$. Let I' contain all i such that $\langle \vec{g}, g \rangle$ does not act regularly on $\Delta_i^{\vec{g},g}$, contain at least one i such that $\langle \vec{g}, g\rangle$ does act regularly on $\Delta_i^{\vec{g},g}$, and be such that the $\tilde{I}$ such that $\coprod_{i \in I'} \Delta_i^{\vec{g},g} = \coprod_{i \in \tilde{I}} \Delta_i^{\vec{g}}$ satisfies that $|\tilde{I} \setminus I|$ is a multiple of the number of copies of the regular $\langle \vec{g} \rangle$ action appearing in the regular $\langle s, \vec{g}\rangle$ action.

Using the strategy from the proof of the density of D_n we can extend s to s' so that $\langle s', \vec{g}\rangle$ acts regularly on $\coprod_{i \in I'} \Delta_i^{\vec{g},g} \setminus \coprod_{j \in I} \Delta_j^{\vec{g}}$. Then $(s', \vec{g}g) \leq_{\mathbb{P}_G} (s, \vec{g})$ and $(s', \vec{g}g) \in E_g$ as was to be shown.

For all $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f\rangle$ is cofinitary and $n \in \mathbb{N}$, the set $H_{f,n}$ is dense in $\mathbb{P}_G$:

Let f, n be as given and $(s, \vec{g}) \in \mathbb{P}_G$. Write $\mathrm{dom}(s) = \coprod_{i \in I} \Delta_i^{\vec{g}}$. Choose a $k \geq n$ such that $k, f(k) \notin \coprod_{i \in I} \Delta_i^{\vec{g}}$ and k and $f(k)$ are not in the same orbit. Let i_0 be such that $k \in \Delta_{i_0}^{\vec{g}}$ and i_i such that $f(k) \in \Delta_{i_1}^{\vec{g}}$. The regular action of $\langle s, \vec{g} \restriction \mathrm{dom}(s)\rangle$ consists of some finite number m of copies of the regular action of $\langle \vec{g} \restriction \mathrm{dom}(s)\rangle$. Choose elements $i_2, \ldots, i_{m-1} \in \mathbb{N} \setminus I$.

Write $\langle s, \vec{g} \restriction \mathrm{dom}(s)\rangle = \coprod_{j<m} \langle \vec{g}\rangle$; the decomposition of $\langle s, \vec{g} \restriction \mathrm{dom}(s)\rangle$ into copies of the regular action of $\langle \vec{g}\rangle$. Choose d_k and $d_{f(k)}$ in $\coprod_{j<m}\langle \vec{g}\rangle$ in different copies of $\langle \vec{g}\rangle$ such that $s(d_k) = d_{f(k)}$. Then define the bijection $F : \coprod_{j<m}\langle \vec{g}\rangle \to \coprod_{m<k} \Delta_{i_m}^{\vec{g}}$ by $F(d_k) = k$, $F(d_{f(k)}) = f(k)$, choosing arbitrary points in the remaining orbits to map to each other and extend according to the $\vec{g}$ action. Then $(s \cup FsF^{-1})(k) = f(k)$, and $(s \cup FsF^{-1}), \vec{g}) \leq_{\mathbb{P}_G} (s, \vec{g})$ as was to be shown.

Define (using the f fixed at the beginning of this section) $\mathcal{D}$ to be

$$\{D_n \mid n \in \mathbb{N}\} \cup \{R_n \mid n \in \mathbb{N}\} \cup \{E_g \mid g \in G\} \cup \{H_{f,n} \mid n \in \mathbb{N}\}.$$

This is a collection of fewer than continuum many dense sets, so, using MA, there exists a filter $\mathcal{G} \subseteq \mathbb{P}_G$ such that $\mathcal{G}$ intersects all these dense sets.

Define $g = \bigcup_{(s,\vec{g}) \in \mathcal{G}} s$. Since $\mathcal{G}$ intersects all D_n and R_n it follows that g is a bijection. Since $\mathcal{G}$ intersects all $H_{f,n}$ we see that $g \cap f$ is infinite.

Now suppose $\langle G, g\rangle$ is not cofinitary. Then there exists a $w(x) \in W_G$ such that $w(g)$ has infinitely many fixed points, but is not the identity. Since $\mathcal{G}$ intersects E_g for all $g \in G$ we can find $(s, \vec{g}) \in \mathcal{G}$ such that $\vec{g}$ contains all elements of G appearing in $w(x)$. We can also assume there is a k such that $w(s)(k)$ is defined and is different from k. Then $w(s)$ is not the identity of $\langle s, \vec{g}\rangle$ and therefore has no fixed points in the regular action of $\langle s, \vec{g}\rangle$. Any extension $(s', \vec{g'})$ of $(s, \vec{g})$ extends s so as to give copies of the regular action of $(s, \vec{g})$, i.e. does not give $w(s')$ fixed points that $w(s)$ does not have. This shows that $w(s)$ and $w(g)$ have the same finite number of fixed points, contradicting the assumption.

We can see in a similar way that $\langle G, g\rangle$ is locally finite: let $\vec{g} \in G$ be finite. Then there exists an s such that $(s, \vec{g}) \in \mathcal{G}$. For all extensions $(s', \vec{g'})$ of $(s, \vec{g})$ the action of $(s', \vec{g})$ outside $\mathrm{dom}(s)$ is isomorphic to the regular action of $(s, \vec{g})$. This shows that $(g, \vec{g})$ is isomorphic to $(s, \vec{g})$ which is a finite group.

§6. The construction of group generators. In this section we prove that there exists a uniformly computable sequence $\langle g_i \mid i \in \mathbb{N}\rangle$ of permutations that generate a cofinitary group whose isomorphism type is not computable.

For the remainder of this article we will fix an arbitrary $A \subseteq \mathbb{N}$ that is Δ^0_2 ($A \leq_T \emptyset'$) but not computably enumerable. We will construct $\langle g_n \mid n \in \mathbb{N}\rangle$; a uniformly computable sequence of elements of $\mathrm{Sym}(\mathbb{N})$ such that

- $G = \langle \{g_n \mid n \in \mathbb{N}\}\rangle$, the group generated by $\{g_n \mid n \in \mathbb{N}\}$, is cofinitary, and
- the following equivalence holds

$$(*) \qquad i \in A \quad \Leftrightarrow \quad \exists g \in G \; \exists t \in \mathbb{N} \; g^{p_{f(i,t)}} = \mathrm{Id},$$

  where $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a computable bijection and p_n is the n-th prime.

This shows that A is c.e. in the isomorphism type of G. So if G had a computable copy, then A would be computably enumerable, contradicting our assumption on A.

Notation 26. $\vec{x} = \langle x_n \mid n \in \mathbb{N}\rangle$ is a sequence of variables. Write $\vec{x} \restriction s$ for $(x_0, \ldots, x_{s-1})$. $F(\vec{x})$ is the free group on the generators $\{x_i \mid i \in \mathbb{N}\}$. $F(\vec{x} \restriction s)$ is the free group on the generators $\{x_i \mid i < s\}$. For $R \subseteq F(\vec{x})$ we write $R \restriction s$ for $R \cap F(\vec{x} \restriction s)$. Fix an enumeration $\langle w_i \mid i \in \mathbb{N}\rangle$ of $F(\vec{x})$.

Definitions 27. (i). We call $\vec{R} = \langle R_s \subseteq F(\vec{x}) \mid s \in \mathbb{N}\rangle$ a *demure*[1] *sequence (of relators)* iff $\vec{R}$ satisfies requirements R1, R2, and R3.

R1 for all $t \in \mathbb{N}$, the set $R_s \restriction t$ is finite.

R2 for all $t \in \mathbb{N}$, $R_{s+1} \restriction t \subseteq R_s \restriction t$.

Let $G_s = G(\vec{R})_s$ be the group $F(\vec{x})/R_s = F(\vec{x})/(R_s^{F(\vec{x})})$, where $R_s^{F(\vec{x})}$ is the normal subgroup of $F(\vec{x})$ generated by R_s. Let $W_{s,\mathrm{Id}} = W(\vec{R})_{s,\mathrm{Id}}$ be the set of words in $F(\vec{x})$ representing the identity in G_s.

R3 for all $t \in \mathbb{N}$, $W_{s,\mathrm{Id}} \restriction t = (R_s \restriction t)^{F(\vec{x} \restriction t)}$.

Note that since $R_s \restriction t$ stabilizes for every t we can define the group $G_\omega = G(\vec{R})_\omega$ as follows: Define $R_\omega := \{w \in F(\vec{x}) \mid \forall t \in$

[1]From answers.com: Modest and reserved in manner or behavior.

$\mathbb{N}\ w \in F(\vec{x} \restriction t) \to \forall k \in \mathbb{N}\ \exists l \geq k\ w \in R_l \restriction t\}$ (this set is the union of all $R_s \restriction t$ for s large enough that the set has stabilized). Then set $G_\omega := F(\vec{x})/R_\omega$ (this group is the inverse limit of the groups G_s). Note that $G_\omega \restriction t = G_s \restriction t$ for s large enough (large enough is so that $R_s \restriction t$ has stabilized). Also note that this means that if $W_{\omega,\mathrm{Id}} = W(\vec{R})_{\omega,\mathrm{Id}}$ is the set of words in $F(\vec{x})$ representing the identity in G_ω, then $W_{\omega,\mathrm{Id}} \restriction t = W_{s,\mathrm{Id}} \restriction t$ for s large enough.

(ii). For $w_1, w_2 \in F(\vec{x})$ write $w_1 \sim_s w_2$ iff w_1 and w_2 represent the same group element in G_s (this is equivalent to $w_1 w_2^{-1} \in W_{s,\mathrm{Id}}$).

(iii). For sequences $\vec{p}$ and $\vec{q}$ of length ω of finite injective functions $\mathbb{N} \to \mathbb{N}$ with finite support and $w \in F(\vec{x})$, we call $\vec{q}$ an *s-good extension* of $\vec{p}$ with respect to w if for all $i \in \mathbb{N}$ we have $p_i \subseteq q_i$ and if $w(\vec{q})(l) = l$ then there are $u_1, u_2, z \in F(\vec{x})$, $m \in \mathbb{N}$ and $z' \in F(\vec{x})$ such that
   (a) u_1, u_2, and z are subwords of w,
   (b) $w = u_1^{-1} z u_2$ without cancellation,
   (c) $u_1 \sim_s u_2$,
   (d) $z \sim_s z'$,
   (e) $z'(\vec{p})(m) = m$, and
   (f) $u_2(\vec{q})(l) = m$.

(iv). We call z an *s-conjugate subword* of w iff there are $u_1, u_2 \in F(\vec{x})$ such that z, u_1, u_2, w satisfy (iii)a, (iii)b, and (iii)c above.

(v). We say $\vec{p}$ *satisfies* $W_{s,\mathrm{Id}}$ iff for all $w \in W_{s,\mathrm{Id}}$ the partial map $w(\vec{p})$ is the identity where defined.

With these definitions we can formulate and prove the next lemma. This lemma shows exactly what we need to achieve in using this method to construct a group of a given isomorphism type.

Lemma 28. *Let $\vec{R}$ be a demure sequence, and let $\langle (\vec{p})_s \mid s \in \mathbb{N} \rangle$ be such that*

- *for all $s \in \mathbb{N}$ $(\vec{p})_s$ satisfies $W_{s,\mathrm{Id}}$,*
- *for all $s \in \mathbb{N}$ $(\vec{p})_{s+1}$ is an s-good extension of $(\vec{p})_s$ with respect to all words in $\{w_0, \ldots, w_s\}$ and their subwords, and*
- *for all $i \in \mathbb{N}$ $g_i := \bigcup_{s \in \mathbb{N}} (p_i)_s$ is a permutation of $\mathbb{N}$.*

Then the group generated by $\langle g_i \mid i \in \mathbb{N} \rangle$ is cofinitary and isomorphic to G_ω.

Proof. We need to see that for all $w \in F(\vec{x})$, if $w \in W_{\omega,\mathrm{Id}}$, then $w(\vec{g}) = \mathrm{Id}$, and if $w \notin W_{\omega,\mathrm{Id}}$, then $w(\vec{g})$ has finitely many fixed points.

It is immediate that if $w \in W_{\omega,\mathrm{Id}}$ then $w(\vec{g})$ is the identity, since $w((\vec{p})_s) = \mathrm{Id}$ for all $s \in \mathbb{N}$.

So let $w \notin W_{\omega,\mathrm{Id}}$ and suppose u is least such that $w \in F(\vec{x} \restriction u)$. Then there is a $t \in \mathbb{N}$ such that $w \notin W_{t,\mathrm{Id}}$. Let Z_s be the set of s-conjugate subwords of w (note that this set does not depend on $\vec{p}$ and $\vec{q}$). Z_s is a finite set, and $Z_{s+1} \subseteq Z_s$. Therefore we can define $Z = \lim_{s\in\mathbb{N}} Z_s$. By induction we assume that for all $z \in Z$, $z(\vec{g})$ has finitely many fixed points. This is reasonable since none of these z is in $W_{\omega,\mathrm{Id}}$ (otherwise w would be in $W_{\omega,\mathrm{Id}}$). Let $t' > t$ be such that for every $z \in Z$ the number (and therefore the set) of fixed points of $z((\vec{p})_{t'})$ is the same as $z(\vec{g})$ and $Z = Z_{t'}$.

We will show that the number of fixed points of $w(\vec{g})$ is bounded by the sum of the number of fixed points of $z((\vec{p})_{t'})$ for $z \in Z$. Let l be a fixed point of $w(\vec{g})$ that is not a fixed point of $w((\vec{p})_{t'})$, and let t'' be the least number such that $w((\vec{p})_{t''})(l) = l$.

Then since $(\vec{p})_{t''}$ is a $(t''-1)$-good extension of $(\vec{p})_{t''-1}$ there exist $z \in Z$, $z', u_1, u_2 \in F(\vec{x})$ and $m \in \mathbb{N}$ as in the definition of $(t''-1)$-good extension with respect to w. In particular $z'((\vec{p})_{t''-1})(m) = m$. However since $z \sim_{t''} z'$ we know $z'(\vec{g}) = z(\vec{g})$. This means that $z((\vec{p})_{t''-1})(m) = m$, and therefore $z((\vec{p})_{t'})(m) = m$. This shows that the new fixed point comes from a fixed point of $z((\vec{p})_{t'})$, i.e. showing the promised bound. ⊣

Now we need to see that a sequence as in the hypothesis of this lemma can be built.

In the proof of the previous lemma you can see the trace of an idea. This idea is to identify what we already know about $w(\vec{g})$ from $(\vec{p})_s$ for some s. Note that this is likely larger than $w((\vec{p})_s)$. In fact what we already know can be described exactly by

$$\bigcup_{w\sim_s w'} w'((\vec{p})_s).$$

The next definition shows how to extend $\vec{p}$ to $\vec{q}$ such that $\vec{q}$ is everything we already know about $\vec{p} \restriction B \cup \mathrm{supp}(\vec{p})$.

Definition 29. Given $\vec{p}$ we say $\vec{q}$ is obtained from $\vec{p}$ by (B, s)-*applying relations* if

$$(a, b) \in q_i \quad \Leftrightarrow \quad \big(\exists w'[x_i w' \in W_{s,\mathrm{Id}} \wedge w'(\vec{p})(b) = a]\big) \wedge (p_i \neq \emptyset \vee i \in B).$$

Lemma 30 (Applying Relations Lemma). *If $\vec{q}$ is obtained from $\vec{p}$ by (B, s)-applying relations for some finite $B \subseteq \mathbb{N}$, then*

(i). *for all $i \in \mathbb{N}$, $p_i \subseteq q_i$;*
(ii). *if $\vec{r}$ is obtained from $\vec{q}$ by (B, s)-applying relations, then $\vec{r} = \vec{q}$;*
(iii). *if $\vec{p}$ satisfies $W_{s,\mathrm{Id}}$, then so does $\vec{q}$;*
(iv). *$\vec{q}$ has finite support, and for all $i \in \mathbb{N}$ q_i is finite;*
(v). *$\vec{q}$ is an s-good extension of $\vec{p}$ for any w.*

Proof. (i) is immediate since $x_i x_i^{-1} \in W_{s,\mathrm{Id}}$.

For (ii): if $x_j w \in W_{s,\mathrm{Id}}$ and $w(\vec{q})(b) = a$, then there exists a w' such that $x_j w' \in W_{s,\mathrm{Id}}$ and $w'(\vec{p})(b) = a$. This word is obtained from w as follows: for any pair (n_0, n_1) from q_i there is an $x_i w_{(n_0,n_1),i} \in W_{s,\mathrm{Id}}$ such that $w_i(\vec{p})(n_1) = n_0$. Now replace an occurrence of x_i in w by $w^{-1}_{(n_0,n_1),i}$ where (n_0, n_1) is the pair used in determining the evaluation path of b in $w(\vec{q})$ at that occurrence.

For (iii): if $w \in W_{s,\mathrm{Id}}$ is such that $w(\vec{q}) \ncong \mathrm{Id}$, then there is a word w' such that $w' \in W_{s,\mathrm{Id}}$ such that $w'(\vec{p}) \ncong \mathrm{Id}$. This w' is obtained as above.

For (iv): $\vec{q}$ has finite support since its support is contained in the union of B and the support of $\vec{p}$. For all $i \in \mathbb{N}$ q_i is finite since it only has pairs (a, b) in it where $a, b \in \bigcup_{i\in\mathbb{N}} (\mathrm{dom}(p_i) \cup \mathrm{ran}(p_i))$.

For (v): if $w \in W_{s,\mathrm{Id}}$ is such that $w(\vec{q})(n)$ is defined, then there is a word $w' \in W_{G,H}$ such that $w' \in W_{s,\mathrm{Id}}$ and $w'(\vec{p})(n)$ is defined and has the same value. ⊣

The following two lemmas show that we can construct a sequence $\langle(\vec{p})_s \mid s \in \mathbb{N}\rangle$ as used in Lemma 28.

Lemma 31 (Domain Extension Lemma). *If $\vec{p}$ satisfies $W_{s,\mathrm{Id}}$ and i, k are such that $k \notin \mathrm{dom}(p_i)$, then there exists an s-good extension $\vec{q}$ of $\vec{p}$ such that $k \in \mathrm{dom}(q_i)$ and $\vec{q}$ satisfies $W_{s,\mathrm{Id}}$.*

Lemma 32 (Range Extension Lemma). *If $\vec{p}$ satisfies $W_{s,\mathrm{Id}}$ and i, l are such that $\notin \mathrm{ran}(p_i)$, then there exists an s-good extension $\vec{q}$ of $\vec{p}$ such that $l \in \mathrm{ran}(q_i)$ and $\vec{q}$ satisfies $W_{s,\mathrm{Id}}$.*

Since the Range Extension Lemma follows from the Domain Extension Lemma by taking inverses of all words we will only prove the Domain Extension Lemma.

Proof. Let $i \leq n$, $k \in \mathbb{N}$ and $\vec{p}$ satisfy $W_{s,\mathrm{Id}}$. First $(\{i\}, s)$-apply relations to $\vec{p}$ to get $\vec{q}$. If $k \in \mathrm{dom}(q_i)$ we are done by the Applying Relations Lemma, so suppose this does not happen. Choose an l such that $l > \max\{\{k\} \cup \bigcup_{i\in\mathbb{N}} (\mathrm{dom}(q_i) \cup \mathrm{ran}(q_i))\}$.

Let $\vec{r}$ be such that $r_j = q_j$ for $j \neq i$ and $r_i = q_i \cup \{(k, l)\}$. We need to see that $\vec{r}$ satisfies $W_{s,\mathrm{Id}}$ and that $\vec{r}$ is an s-good extension of $\vec{q}$. We will first show that $\vec{r}$ satisfies $W_{s,\mathrm{Id}}$, so suppose that there is a $w \in W_{s,\mathrm{Id}}$ such that $w(\vec{r}) \ncong \mathrm{Id}$ and let w be the shortest such word. Since $w(\vec{q}) \cong \mathrm{Id}$, this new computation $w(\vec{r})(a) \neq a$ uses the pair (k, l). Since $l > \max\{\bigcup_{i\in\mathbb{N}} (\mathrm{dom}(q_i) \cup \mathrm{ran}(q_i))\}$ this pair has to be used at the beginning or the end; if it is used in a location in the middle then it needs to be used again immediately (in the opposite direction). This would mean w has a subword $x_i^{-1} x_i$ or $x_i x_i^{-1}$ contradicting its minimality. If (k, l) is used at the beginning and the end then $w = x_i w' x_i^{-1}$, $l = a$ and $w(\vec{r})(a) = a$ contrary to the assumption that a is not a fixed point. So (k, l) is used only once either at the beginning of the word or at the end of the word. This however contradicts $\vec{q}$ having been obtained by applying relations; if there is a word w such that $w = w' x_i$ and $w'(\vec{q})(k)$ is defined, then $(k, w'(\vec{q})(k)) \in q_i$ (for exact correspondence with the definition of applying relations consider $x_i(w')^{-1}$), showing that k was already in the domain of q_i.

To see that $\vec{r}$ is an s-good extension of $\vec{q}$ let $w \in F(\vec{x})$ be minimal such that $w(\vec{r})(a) = a$ and $w(\vec{q})(a) \uparrow$. By minimality, as in the previous paragraph, the pair (k, l) can only be used at the beginning or the end. Since $w(\vec{r})(a) = a$ if it is used at the beginning or the end, then $a = l$. This implies that it also must be used at the end or the beginning, which implies that $w = x_i w' x_i^{-1}$ and $w'(\vec{q})(x_i^{-1}(l)) = x_i^{-1}(l)$ showing that we have an s-good extension. ⊣

It is now clear how to construct a sequence of generators $\langle g_i \mid i \in \mathbb{N}\rangle$ generating a given group G. We must find a demure sequence $\vec{R}$ such that for this sequence we have that G_ω satisfies (*) (see page 15). Then by alternately applying the Domain and Range Extension Lemmas we construct a sequence $\langle (\vec{p})_s \mid s \in \mathbb{N}\rangle$ satisfying the requirements in Lemma 28 and giving rise to the group we need.

It is clear from the proof of the Domain and Range Extension Lemmas that if the applying relations operation is computable, then domain and range extension are computable. Therefore we need to come up with a demure sequence such that (B, s)-applying relations is computable for any finite B and s. For this it is sufficient to be able to find the finite set $R_s \upharpoonright t$ effectively, for all s and t, if $\vec{p}$ and B are given. Let $F = B \cup \operatorname{supp}(\vec{P})$. Let $t = \max F + 1$. Then $(F \upharpoonright t) \cap R_s$ can be computably found. Therefore we can effectively apply relations with only the words in this finite set. We see this is sufficient as follows: since the sequence is demure $W_{s,\mathrm{Id}} \upharpoonright t = (R_s \upharpoonright t)^{F(\vec{x}\upharpoonright t)}$. That is any $W_{s,\mathrm{Id}}$ is obtained from $R_s \upharpoonright t$ by conjugating by x_i with $i \leq t$ and taking products. It is easy to see that if $x_i w_0 w_1 \in W_{s,\mathrm{Id}}$, $w_1 \in W_{s,\mathrm{Id}}$, and $w_o w_1(\vec{p})(b) = a$, then also $w_0(\vec{p}(b) = a$, and if $x_i w x_i^{-1} \in W_{s,\mathrm{Id}}$ has $w x_i^{-1}(\vec{p})(b) = a$ then already $x_i(\vec{p})(a) = b$; showing that is it sufficient to only consider the relators.

We have a set A that is Δ^0_2 but not computably enumerable. Since it is Δ^0_2 there exists a function $h : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that $\xi_A(n) = \lim_{s\to\infty} h(n, s)$.

Define R_s to be

$$\{x_{f(i,t)}^{p_{f(i,t)}} \mid i > t \ \vee\ \forall j(t \leq j \leq s \to h(i, j) = 1)\},$$

where $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a computable bijection. It is easy to see this is a uniformly computable demure sequence: the computability is clear from the definition. R1 is satisfied since for every $i < t$ the set $R_s \upharpoonright t$ contains at most one relator of the form x_i^k where $k \in \mathbb{Z}$. R2 is satisfied since $i > t \ \vee\ \forall j(t \leq j \leq (s+1) \to h(i, j) = 1)$ implies $i > t \ \vee\ \forall j(t \leq j \leq s \to h(i, j) = 1)$. Finally R3 is satisfied since there are no relators relating x_i with x_j for $i \neq j$.

Then R_ω is the set

$$\{x_{f(i,t)}^{p_{f(i,t)}} \mid \forall j \geq t\ h(i,j) = 1\}.$$

Write $E := \{(i,t) \in \mathbb{N} \times \mathbb{N} \mid (\forall j \geq t)\ h(i,j) = 1\}$. Thus $(i,t) \in E$ if $\lim_{s\to\infty} h(n,s) = 1$ and $h(n,s)$ has converged before or at stage t. So $n \in A$ iff there is a t such that $(n,t) \in E$, iff there is a t such that $x_{f(n,t)}^{p_{f(n,t)}} \in R_\omega$, iff there is a generator g for G_ω and $t \in \mathbb{N}$ such that $g^{p_{f(n,t)}} = \mathrm{Id}$. This is already fairly close to (*), but we need it to be true not just for the generators but for the whole group. For this note that

$$G_\omega \cong (*_{(i,t)\in E}\mathbb{Z}_{f(i,t)}) * (*_{i\in\mathbb{N}}\mathbb{Z}),$$

where $G * H$ is the free product of the groups G and H, and $*_{i\in\mathbb{N}}G_i$ the infinite free product of the groups G_i.

Theorem 33. *Each element of a free product is conjugate to a cyclically reduced element.*

For this see for instance [LS, Thm. 1.4, Chap IV] (the theorem there is stated for the free product of two groups, but it easily generalizes to our current situation).

Now if $w \in G_\omega$ is a cyclically reduced word of length greater than 1, then clearly w has infinite order. This shows that the only elements of G_ω that have finite order are conjugates of generators, and for those we had already shown that they have the appropriate orders. This completes the proof.

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